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Chapter 1

Introduction

The idea of the production process as a circular flow was typical of the political economy of the 18-th and 19-th century.[1] This point of view considers that the great majority of the commodities that we find in an economy necessitates of many other commodities to be produced. Many commodities, therefore, are, at the same time, products for some sector of production and means of production for some other. Moreover, in a closed economy, the various industries can acquire their means of production only by means of the revenues that they have obtained from selling their output. The complex inter-industrial, technical relationships that characterize an economy become central in the study of fundamental economic phenomena, as the determination of prices or the distribution of income, which cannot be determined meaningfully through partial equilibrium analysis.

However, following the *marginalist revolution* that took places at the beginning of the last century this point of view has been replaced by the one of a "one way avenue that leads from 'Factors of Production' to 'Consumption Goods' " Sraffa (1960). The starting point of most neoclassical analysis, indeed, are the *given* endowments that can be found in an economy. Those endowments make in motion the whole production process, which eventually ends up in some final goods whose prices represent the relative scarcity of the factors used for its production. For most classical economists, on the other hand, the scarcity of natural resources was only a special case with respect to the majority of factors of production, that are *reproducible*.

In the first part of this dissertation we will analyse how the idea of the production process as a circular flow has been considered by three of the major thinkers in the economic field of the century: Wassily Leontief, Piero Sraffa and John Von Neumann.

Leontief's foundation of input-output approach, indeed, is based on the analysis of the inter-sectoral exchanges that characterize an economic system, in which the same commodity is seen at the same time as an input and as an output, and has been inspired by the tradition of classical economy.

The reference to classical economists is even more explicit in Piero Sraffa's *Production*

[1] Among the main representatives of this tradition we find François Quesnay, Adam Smith, David Ricardo, John Stuart Mill and Karl Marx.

of Commodities by Means of Commodities, which is intended, as the author writes, as a critique to the marginalist approach. The center of Sraffa's analysis is indeed the relationship between the exchange values of commodities that needs to be determined to guarantee the reproducibility of the economic system and the distribution of the value of the surplus generated by the production process.

Finally, Von Neumann's analysis of a quasi-dynamic General Equilibrium also presents an economy characterized by an heterogeneity of commodities which are completely employed in the production of other commodities, in absence of scarce natural resources or initial endowments.

In the second part of the dissertation, on the other hand, we proceed to the construction of an Agent-Based Model based on the considerations made in the first part. We build other model by steps, progressively introducing further complications in order to analyse the different effects separately.

We start from a "purely physical" model, in which there is no price system and no heterogeneity among agents. This simple system will allow us to reason on the technical and physical necessities that an economic system has in order to being able to reproduce itself and grow.

We then consider a model in which there are two social classes which, in some way, compete over the surplus generated, still in absence of a price system. In a first version we analyse the simple case in which one social class, the workers, receives a fixed amount of resources per capita every year and the other, the firms, keeps the residual. We then analyse how things change if the resources received as wage by workers change every period, being a fixed or variable proportion of the surplus generated in the current period. We finally analyse the effects of some rigidities in the determination of wage, when the current level of resources going to the workers depend of the surplus generated in the previous period.

Finally we introduce the determination of prices, on the basis of the analysis done by Sraffa. We will therefore be able to reason on some theoretical results in a more intuitive way, as well as to analyse the evolution of aggregated macro-variables in a growing economy.

Part I

Theoretical framework

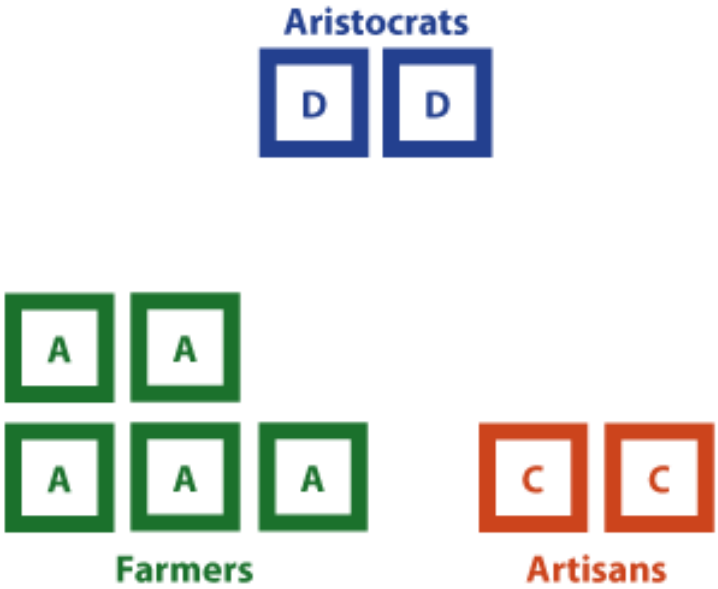
Chapter 2

The Tableau économique

In the *Tableau économique* we meet for the first time two important concepts that would influence the study of economic matters for a long time.
The first one is the idea of a "surplus", meaning the idea that in an economy may be able to produce more than what it is strictly necessary for the reproduction of the production process. In the 18th century, the french economist François Quesnay sees the origin of the creation of such a surplus in the physical properties of Nature. Land, indeed, is able to produce, from a certain amount of a product, a greater amount of the same product. For this reason the school of economic thinking founded by Quensnay received the name of Physiocracy (from the greek *physis*, natura, and *kratos*, power).
The other fundamental idea is the vision of the production process as a circular one: different production sectors need commodities produced in other production sectors in order to start over the production each year, and they can pay for it with the revenues they obtain from selling their production. This is the idea that we will explore in this dissertation, through the analysis of various authors, starting right from Quesnay.

The economy presented by the *Tableau économique* is an extremely simply one. The population is divided in social classes, according to their role in the production process and their legal rights: we have a Productive Class, the farmers, that, as mentioned before, are the only class that is able to increase the value of its production with respect to the means of production employed; a Sterile Class, the artisans, which simply transform the raw material they buy into crafts, which nevertheless will have the same value of the raw material they are produced from; and finally a class of Rentiers, the aristocracy, which do not produce anything but owns the land, so that farmers have to pay rent to them.
The functioning of the economy as described by the Tableau consists in the following process. Let us start looking at the production process. The farmers produce agricultural raw materials for a value of 5 milliards, using two-milliard-worth of agricultural products and one-milliard-worth of crafts. The artisans produce crafts for the value of two milliards, using two-milliard-worth of agricultural raw materials. The aristocracy

does not produce anything, but possesses two milliards in money from previous year rent. The situation at the end of the production process is described by the following image. Each square represent one milliard of value, which can be in agricultural goods (green), crafts (red) or money (blue)



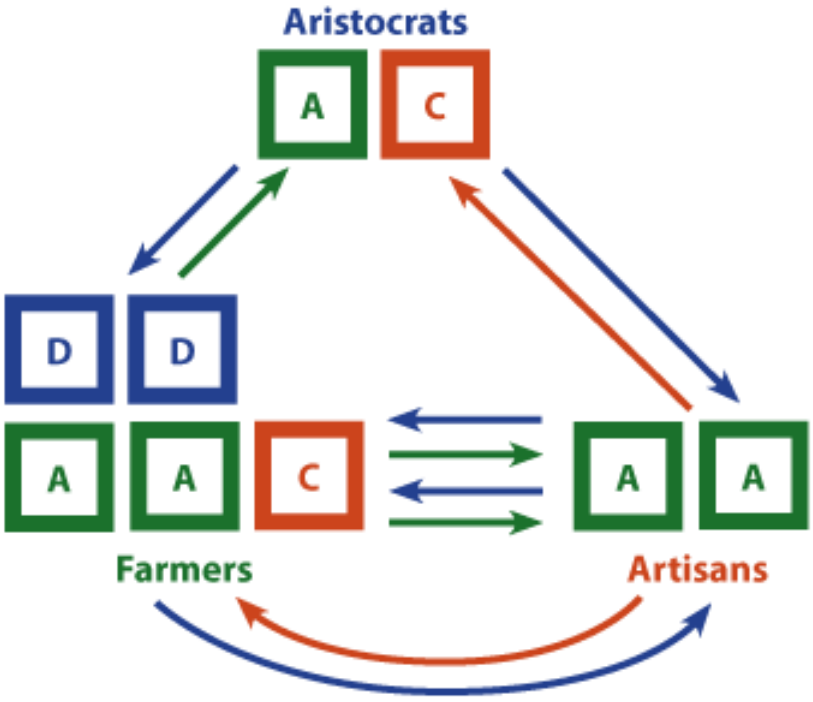
Then let us consider what happens after the production process, when agents need to obtain the necessary commodities in order to produce again, survive and, in general, consume.

The farmers retain raw materials for the value of two milliards in order to use them next year; they need also one-milliard-worth of crafts in order to produce, that they acquire from the artisans; moreover, they sell one milliard-worth of agricultural products to aristocrats and two to artisans. They finish the exchange period with two milliards in money, two milliards in agricultural product and one milliard in craft.

The artisans sell everything they have produced, one-milliard-worth to aristocrats and one-milliard-worth to farmers. With their revenues, they buy two-milliard-worth of agricultural products from farmers.

The aristocrats spend all their money, buying one-milliard-worth of agricultural products and one-milliard-worth of crafts.

The situation after the exchanges have taken place is depicted in the following image. Notice that the arrows indicates the transfers of commodities / money , so that all the green arrows originates from the farmers and all the red arrows originates from the artisans.



Once the farmers have paid two milliards of rents to the aristocrats, the situation has returned exactly equal to the one preceding the production, that in this way can take place once again.
As we can see the economy described in the tableau is a very simple one, but nevertheless is able to underline the interdependencies which exists in an economic system, and the role of distribution in determining the possibility of reproduction.

Chapter 3

Wassily Leontief and input-output analysis

3.1 Introduction

Wassily Leontief was a Russian-American economist who is considered the founder of that area of economics denoted *input-output analysis*. For this he was awarded with the Nobel Price in 1973.[1]

The input-output method is strongly influenced by the classical tradition that sees the economy as a reproduction system. He himself admitted this influence stating that "If [in my model] there is an influence, it is mainly the one of classical economists" (Leontief and Rosier, 1986), and mentioned Marx and Sismondi as some of the most influential authors in the origin of his thought (quoted in Clark (1984)). On the other hand, the compatibility of Leontief's input- output approach with the General Equilibrium of Leon Walras (and, more in general, with the neocalssical framework) is debatable: the neoclassical school, in particular Samuelson, tends to see Leontief's models as particular cases of Walras', while other schools of though (in particular the Neo-Ricardian one) affirm that the description of the economy as a circular flow, in which proper "factors of production" do not actually exist, is a sufficient condition for placing Leontief outside neoclassical tradition.

In any case, Leontieff developed his method of analysis as a response to the weakness he saw in Marshallian partial equilibrium analysis, derived by his econometric work on partial equilibrium supply and demand curves (Akhabbar and Lallement, 2010). He was also a strong critic of the subjectivist approach to economic, according to which economic phenomenon are explained by individual behaviour based on psychological assumptions. In his 1928 dissertation he affirms that "[..] the circular flow principle seems to deserve a higher logical rating than the extremely complex notion of economic man".

His work in the construction of its Input-Output table from empirical data is described by the author himself as the attempt of building a *Tableau économique* of the USA for a particular year (Leontief, 1936). From that table, which gathers the value of

[1] Prize motivation: "For the development of the input-output method and for its application to important economic problems".

all commodities produced by the various industries in the economy, seen both as outputs of such industries and inputs of other industries, he will build a theoretical representation of the economy as a circular flow, imposing the fundamental assumption of constant return to scale.

In the following pages we will follow its steps, describing initially the conceptual building of an Input-Outpu tabel, and then presenting Leontief's models, which traditionally are distinguished in two categories: Closed Models and Open Models.

3.2 The Input-Output Table

A fundamental tool in order to analyze the interdependence and circular nature of the economic system is represented by the Input-Output table. In this first analysis, let us consider it just a descriptive tool to underline the relationship between the different sectors of the economy. We take therefore everything as given: price, quantities, incomes. The Input-Output Table is therefore just a description of an economy in a particular model.

Let us consider an economy composed of n different industries. Let Q_i, $i = 1, ..., n$, denote the physical quantity of commodity i produced by the correspondent industry and p_i, $i = 1, ..., n$, denote the price of that commodity. Moreover, denote with q_{ij} the quantity of commodity i that is employed annually in the production of commodity j. Collect now the n prices into the price vector $\mathbf{p}$ and the n^2 quantities q_{ij} into the matrix $\mathbf{Q}$. Consider the matrix that results from the product of $\mathbf{p}$ with $\mathbf{Q}$:

$$\mathbf{p}'\mathbf{Q} = \begin{bmatrix} q_{11}p_1 & q_{12}p_1 & \cdots & q_{1n}p_1 \\ q_{21}p_2 & q_{22}p_2 & \vdots & q_{2n}p_2 \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}p_n & q_{n2}p_n & \cdots & q_{nn}p_n \end{bmatrix} \tag{3.1}$$

The matrix 3.1 summarizes the technical relationship between all industries in the economy. In each row of the matrix we can follow the use of a specific commodity: each element of the $i - th$ row, in fact, represents the monetary value of the amount of commodity i employed in the production of all the commodities of the economy (commodity i included). On the other hand, each column of matrix 3.1 represents a specific industry: each element of the $j - th$ columns represents the monetary amount that the industry producing commodity j spends for acquiring each commodity it needs.
We still lack some elements in order to complete our Input-Output table. First of all, nothing says that the inter-industrial exchanges represented in matrix $\mathbf{p}'\tilde{\mathbf{A}}$ exhaust all the commodities produced by the economy. The difference between the amount of commodities produced and the amount of commodities employed in the production is a technical matter, but in general every observed modern economy has the technical capabilities of producing a *surplus*.

Table 3.1: Input-Output Table

Input	Output					
	Industry 1	Industry 2	...	Industry n	Final Sector	
					Cons.	Inv.
Commodity 1	$q_{11}p_1$	$q_{12}p_1$	...	$q_{n1}p_1$	c_1p_1	k_1p_1
Commodity 2	$q_{21}p_2$	$q_{22}p_2$	...	$q_{2n}p_2$	c_2p_2	k_2p_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
Commodity n	$q_{n1}p_n$	$q_{n2}p_n$	...	$q_{nn}p_n$	c_np_n	k_np_n
Final Sector						
Wages	L_1w	L_2w	...	L_nw		
Other	r_1	r_2	...	r_n		

We have therefore to add to $\mathbf{p'Q}$ a further column, representing the *Final Sector*, which comprehends the monetary value of this surplus, or the *Net Product* of the system. Specifically, the $i-th$ element of this $(n+1)-th$ column represents the monetary value of that part of the production of the $i-th$ commodity that is not used as mean of production this year. A greater specification may include the distinction between the possible uses of that surplus. For instance, we may distinguish which part of the surplus has been consumed and which part has been employed as investment for the increase of next year means of production. We can therefore denote by c_i, $i = 1, .., n$ that part of the surplus of the $i-th$ commodity that has been consumed and with k_i that part that has been used as investment.
At the same time, expenditures in each industry are not limited to the acquisition of commodities as means of production. We have to consider also the remuneration of all the individuals which participated in the process of production. This is usually called the *added value* of production, since it adds up over the cost of commodity to determine the final value of the production of an industry. Also in this case we could find several kinds of remunerations, but following Leontief (1974) we will consider only labor remuneration and aggregate all other kinds of retribution. We denote with L_i the amount of labor employed annually in the production of commodity i, w the unit wage and r_i the monetary value of the residual value added in production of commodity i. Of course , at the aggregate level, the total value added in the economy will equal the Net Product of the system, so that the sum of the elements of the last two columns will be equal to the sum of the elements of the last two rows.
We can see a complete description of the interdependencies of our economy in Table 3.1, our Input-Output Table.

A simple example will help clarify the construction of the Input-Output Table. Con-

Table 3.2: Flow of commodities in physical terms

	Wheat Industry		Iron Industry		Pigs Industry		
Wheat	240q	+	90q	+	120q	=	450q
Iron	12t	+	6t	+	3t	=	21t
Pigs	180p	+	120p	+	300p	=	600p
	↓		↓		↓		
Production	450q		21t		600p		

sider a simple three-commodities economy producing wheat, iron and pigs.[2]Suppose the economy is in a fully stationary state: all the net product is consumed (so we will have only one column in the Final Sector), and the economy replicates itself identically years after years. Let us suppose, moreover, that all the consumed Net Product goes to the workers, so that we do not have to worry for the heterogeneity of retributions. At first, let us "consider man like horses" and count their consumption into the necessary quantities that an industry has to buy to activate the production process, exactly like gasoline for machines. In this way we are basically ignoring the "Final Sector" mentioned before, and considering only the technical relationships between industries in an economy that reproduces itself. Suppose that the *observed* flows of commodities among industries are the ones summarized by Table 3.2.

Notice that while we can sum all the elements in the rows, since they are measured in the same unit of measure (quarters for wheat, tons for iron and units for pigs), we cannot do the same for the columns, since the means of production of each industry are heterogeneous.

Now let us add our fourth column and row, the Final Sector. To do so, suppose that in the wheat industry are employed 18 workers, 12 in the iron industry and 30 in the pigs industry (in total, 60 employed). Suppose, moreover, that a worker consumes, on average, three quarters of wheat and 5 pigs per year. In order to consider the Final Sector we have to add a row at the bottom counting the number of workers employed in each industry. In doing so, we have to take out from the means of production used in each industry the amount of commodities that were used to sustain workers (since they are now replaced by the workers themselves). For example, in the wheat industry, we have to subtract from means of production reported in table 3.2 54 quarters of wheat (18 workers times 3 quarters) and 90 pigs (18 workers times 5 pigs). And the same applies for the iron and pigs industries. At the same time, the consumption of workers can now be listed in the column of the Final Sector, and amount to 180 quarters of wheat and 30 pigs. We can see all these operations in Table 3.3.

Notice that since this is just a reformulation over *observed* quantities, obviously, for accounting reason, the total amount of commodities produced will equal the total amount

[2]The example comes from Sraffa (1960) cap 1, modified by Pasinetti (1975)

Table 3.3: Flow of physical commodities and labor

	Wheat Industry		Iron Industry		Pigs Industry		Final Sector		
Wheat	186q	+	54q	+	30q	+	180q	=	450q
Iron	12t	+	6t	+	3t	+	0	=	21
Pigs	90p	+	60p	+	150p	+	300p	=	600p
Final Sector	18m	+	12m	+	30m				
	↓		↓		↓				
Production	450q		21t		600p				

Table 3.4: Flow commodities and labor in monetary terms

	Wheat Industry		Iron Industry		Pigs Industry		Final Sector		
Wheat	18.6	+	5.4	+	3	+	18	=	45
	+		+		+				
Iron	12	+	6	+	3	+	0	=	21
	+		+		+				
Pigs	45	+	30	+	75	+	150	=	300
	+		+		+				
Final Sector	9.9	+	6.6	+	16.5				
	=		=		=				
Production	45		21		300				

of commodities employed or consumed.

Let us now conclude this example and compute the Input-Output table in the form of Table 3.1. We have therefore to multiply each physical quantity for its price. Without anticipating the process of determination of prices, let us say that the *observed* exchange ratios are 10 quarters of wheat : one ton of iron : 20 pigs : 1.1818 years of man-power. Using the iron as *numeraire*, we can translate Table 3.3 into monetary values,[3] as in Table 3.4. Notice that now we were able also to sum the columns, since we expressed everything in the same unit of measure.

Notice that it is now clear that also the very simple economy of the *Tableau économique* presented in the previous chapter can be represented in an input-output table (Phillips, 1955). Considering the aristocratic class as an "industry" supplying "renting services" we can summarize Figures 3.5 with just a simple table, Table

Let us come back at our generic Table 3.1 and try to find a more practical and

[3]Unit prices: $p_g = 0.1$, $p_i = 1$, $p_p = 0.5$, $w = 0.55$

Table 3.5: tableau

	Farmers		Artisans		Aristocrats		
Agricultural goods	2	+	2	+	1	=	5
	+		+		+		
Crafts	1	+	0	+	1	=	2
	+		+		+		
Renting service	2	+	0	+	0	=	2
	=		=		=		
Production	5		2		2		

tractable way to present the interdependencies of the production process. In order to do so, let us simplify further the notation, and let us denote the Final Sector as the other industries.[4]Moreover, for notation simplicity, consider $n+1=m$. As we have said, the sum of the elements of each row coincides with the sum of the elements of each column (they are both equal to $Q_i p_i$, for each row i and each column i). Therefore we can present Table 3.1 (with the mentioned modifications) into the following two systems of identities

$$\begin{cases} q_{11}p_1 + q_{12}p_1 + \cdots + q_{1m}p_1 \equiv Q_1 p_1 \\ q_{21}p_2 + q_{22}p_2 + \cdots + q_{2m}p_2 \equiv Q_2 p_2 \\ \quad \vdots \\ q_{m1}p_m + q_{m2}p_m + \cdots + q_{mm}p_m \equiv Q_m p_m \end{cases} \tag{3.2}$$

$$\begin{cases} q_{11}p_1 + q_{21}p_2 + \cdots + q_{m1}p_m \equiv Q_1 p_1 \\ q_{12}p_1 + q_{22}p_2 + \cdots + q_{m2}p_m \equiv Q_2 p_2 \\ \quad \vdots \\ q_{1m}p_1 + q_{2m}p_2 + \cdots + q_{mm}p_m \equiv Q_m p_m \end{cases} \tag{3.3}$$

We can, anyway, rearrange systems 3.2 and 3.3 in an even more convenient way. We define the *production coefficents*

$$a_{ij} = \frac{q_{ij}}{Q_j} \qquad i, j = 1, \ldots, n \tag{3.4}$$

The single element a_{ij}, therefore, represents the amount of commodity i necessary for the production of *one unit* of commodity j by the correspondent industry. We can therefore

[4]This means to aggregate, for every i, c_i and r_i into q_{in+1}, and, for every j, $l_j w$ and r_j into $q_{(n+1)j}p_{n+1}$, where p_{n+1} is the price of the good produced in this sector (if we consider only labor, the wage).

re-write systems 3.2 and 3.3 in the following form[5]

$$\begin{cases} a_{11}Q_1 + a_{12}Q_2 + \cdots + a_{1m}Q_m \equiv Q_1 \\ a_{21}Q_1 + a_{22}Q_2 + \cdots + a_{2m}Q_m \equiv Q_2 \\ \quad \vdots \\ a_{m1}Q_1 + a_{m2}Q_2 + \cdots + a_{mm}Q_m \equiv Q_m \end{cases} \tag{3.5}$$

$$\begin{cases} a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \equiv p_1 \\ a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \equiv p_2 \\ \quad \vdots \\ a_{1m}p_1 + a_{2m}p_2 + \cdots + a_{mm}p_m \equiv p_m \end{cases} \tag{3.6}$$

3.3 Leontief Closed Model

Notice, once again, that the systems 3.5 adn 3.6 are systems of *identities*, resulting from accounting classification of *observed* quantities and price. If we want to consider them as systems of *equations* we have to make some assumption regarding the production techniques or the distribution process. A fundamental assumption usually made by Leontief is the Constant Return of Scale assumption. This means that increasing by a factor the inputs employed in the production, the output will increase of the same factor. To make an example, if I need two quarters of wheat to produce 1 pigs, I will need 4 quarters of wheat to produce two pigs, and the same applies for every amount of commodity used to produced other commodities. An important consequence of this is that the coefficients of production remain constant following a variation of the price vector $\mathbf{p}$ and of the vector gathering the total quantities produced by each sector of the economy, i.e. the *total scale* at which each industry is operating, vector $\mathbf{q}$. In this way systems 3.5 and 3.6 do not represent any more a description of the economy, but a specific *theory* that can be accepted or rejected.[6]If we make the assumption of constant return to scale, then the systems 3.5 and 3.6 become the following systems of equation, expressed

[5]For system 3.2 divide the $i-th$ equation for the correspondent price and substitute q_{ij} with $a_{ij}Q_j$, by definition 3.4. For system 3.3 divide the $i-th$ equation for the correspondent total quantity and apply definition 3.4

[6]Notice that the assumption of constant return to scale is not the only one possible. It could be possible also, say, to consider prices as quantities as given, and the matrix of coefficients as unknown. The distinction between given variables and unknown variables depend on the information at the disposal of the economist and, even more, on the question the economist is asking. As Leontief writes "The decision [..] is essentially a tactical one. The theoretical formulation is a weapon." (Leontief, 1974)

in matrix notation

$$\begin{bmatrix} a_{11}-1 & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22}-1 & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm}-1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.7}$$

$$\begin{bmatrix} p_1 & p_2 & \dots & p_m \end{bmatrix} \begin{bmatrix} a_{11}-1 & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22}-1 & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm}-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix} \tag{3.8}$$

or, calling $\tilde{A}$ the (m,m) matrix of technical coefficients

$$(\tilde{\mathbf{A}} - \mathbf{I})\mathbf{q} = \mathbf{0} \tag{3.9}$$

$$\mathbf{p}'(\tilde{\mathbf{A}} - \mathbf{I}) = \mathbf{0} \tag{3.10}$$

and constitute the so-called *Leontief Closed Model*. It is called "closed" since the (n+1)-column, the final demand, is considered as any other industry in the economy: the "Household Industry" provides "value added services" to all other industries, the (n+1)-th row, and "employs" other commodities according to consumption decision, the (n+1)-th column (Leontief, 1937). This setting makes sense in the case of a closed economy in a pure stationary state : all the decisions are made identically year after year, everything is consumed, no technological change occurs.

Let us now consider at which conditions systems 3.9 and 3.10 admit solutions. First of all let us consider an implicit property of the matrix $\tilde{\mathbf{A}}$:

$$\tilde{\mathbf{A}} \geq 0 \quad or \quad a_{ij} \geq 0 \quad \forall i, j \tag{3.11}$$

This is obvious recalling the construction of coefficients a_{ij} from physical quantities, that clearly cannot be negative.
Then consider the following proposition

Proposition 3.3.1. *Consider the non-negative matrix* $\tilde{\mathbf{A}}$. *if the following holds*

$$Det[\tilde{\mathbf{A}} - \mathbf{I}] = 0 \tag{3.12}$$

Then there exist infinite, non-zero and non-negative solutions for vectors $\boldsymbol{p}$ *and* $\boldsymbol{q}$

Proof. If 3.12 holds then systems 3.9 and 3.10 are homogeneous systems with singular matrix of coefficients, and therefore have infinite non-zero solutions.

To show the non-negativity of the solution consider that, because of hypothesis 3.12, one root of the characteristic equation associated with $\tilde{\mathbf{A}}$,

$$Det[\lambda\tilde{\mathbf{A}} - \mathbf{I}] = 0 \tag{3.13}$$

is $\lambda = 1$. Therefore 1 is one eigenvalue of $\tilde{\mathbf{A}}$. We can even show that it is the maximum eigenvalue of $\tilde{\mathbf{A}}$: Consider the matrix $\mathbf{Q}$ mentioned at the beginning of this chapter containing all quantities q_{ij}. We constructed matrix $\tilde{\mathbf{A}}$ by dividing each column of $\mathbf{Q}$ for the correspondent Q_j. This is analogous of saying that

$$\tilde{\mathbf{A}} = \mathbf{Q}\mathbf{T}^{-\mathbf{1}} \tag{3.14}$$

where $\mathbf{T}$ is a diagonal matrix with total quantities $Q_1, Q_2, ..., Q_{n+1}$ on the diagonal. Consider instead the matrix $\mathbf{T}^{-\mathbf{1}}\mathbf{Q}$ (which correspond on dividing each *row* of matrix $\mathbf{Q}$ by the correspondent Q_i). By definition the sum of the elements of each row will be 1, since

$$\sum_{i=1}^{n+1} \frac{q_{ij}}{Q_i} = \frac{\sum_{i=1}^{n+1} q_{ij}}{Q_i} = \frac{Q_i}{Q_i} = 1 \tag{3.15}$$

But then Proposition 10.2.9 of Mathematical Appendix guarantees that the largest eigenvalue of $\mathbf{T}^{-\mathbf{1}}\mathbf{Q}$ is equal to one. Since $\mathbf{T}^{-\mathbf{1}}\mathbf{Q}$ is *similar* to $\tilde{\mathbf{A}} = \mathbf{Q}\mathbf{T}^{-\mathbf{1}}$[7] they have the same maximum eigenvalue. Given this result, being $\tilde{\mathbf{A}}$ non negative, Proposition 10.2.3 of Mathematical Appendix guarantees the non negativity of both $\mathbf{p}$ and $\mathbf{q}$.

□

It is worth to spend some words on the economic interpretation of Proposition 3.3.1.

In the first place, what is the economic meaning of the hypothesis 3.12? It means that of the (n+1) equations of system 3.7 at least one must be linearly dependent from the others. Consider the first n columns. They represent the technical process of the various industries, so there is very little economic rationale for them to be linearly dependent. We are left only with the last column, the Final Demand column. The condition 3.12 is telling us is that the Final Demand, the last equation, must be dependent on the productive capacity of the economy, the first n equations. This makes sense, since in a closed system the income cannot come from anywhere than the internal production, the internal technical capacity of generating income, so that the Final Demand necessarily depend on the methods of production. Actually hypothesis 3.12 it is telling us more: that, in this closed stationary model, the final demand must be exactly equal to the production capability of the economy, if we want the system to have non-zero solution. Notice that condition 3.12 does not merely say that final demand cannot be greater than productive capability (clearly impossible in a closed system), but cannot even be smaller, if we want all the equation to be satisfied. If that were not the case, we would have the output of an industry greater than the input it receives and/or the sum of the labor needed by the industries would be smaller than the amount of labor present in the system(Q_{n+1} in this notation). This condition for Pasinetti (1975) can be therefore considered similar to the Keynesian condition that effective demand must be equal of the productive capacity of the system, if we want full employment.

[7] Two matrix $\mathbf{A}$ and $\mathbf{B}$ are similar if there exist an invertible matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-\mathbf{1}}\mathbf{A}\mathbf{P}$. Considering matrix $\mathbf{T}$ as matrix $\mathbf{P}$ of the definition, clearly $(\mathbf{T}^{-\mathbf{1}})(\mathbf{Q}\mathbf{T}^{-\mathbf{1}})(\mathbf{T}) = \mathbf{T}^{-\mathbf{1}}\mathbf{Q}$

Secondly, we may question the economic meaning of the infinity of solutions of the two systems, since it is clear than in an economy only a vector $\mathbf{q}$ and a vector $\mathbf{p}$ are realized. Given that we can easily assume the rank of $\mathbf{I} - \mathbf{A}$ to be n (since, as we have said, there is little economic rationale for the technical coefficients of various industries to be linearly dependent), if we set the value of a variable all the others are consequently uniquely determined.
For the system of prices this has a straightforward economic meaning: in fact what matters are the *relative prices* between commodities, so that one commodity can always be chosen as *numeraire*.
For the quantity system, on the other hand is a bit more complex, since the concept of *relative quantity* does not make much sense. Nevertheless, not all the elements of vector $\mathbf{q}$ are equal: if the first n elements are the quantities produced in the various industries, the last one represents the amount of labor present in the system (Q_{n+1}), a quantity that does not respond only to economic rationale. Therefore it can be, and actually often is, considered exogenous, so that also the system of quantities has a unique solution.

3.4 Leontief Open Model

One of the main shortcoming of the Leontief Closed Model is exactly the fact that it treats consumption and investment decisions as they were production decisions. In substance, as we have mentioned in the previous section, the Final Sector is considered as a regular industry, consuming goods and producing labor (and eventually other value-adding services). In other words, the problem with this approach is that, while the coefficients of the sub-matrix of $\tilde{\mathbf{A}}$ composed by the first n rows and columns originate from the technique of production, the ones of the last column/row have completely different origin: consumption and investment choice, and distribution of the value added, are the fruit of complex economic dynamics, and far more susceptible to change (something that would kill the constant return to scale assumption).
Form this reason Leontief in his later works decided to consider the last column of the matrix $\tilde{\mathbf{A}}$ as *given*, and therefore to be determined outside the quantity and price decision, according to other criteria (Pasinetti, 1975). The question posed by what has been called the *Open Leontief Model* can therefore be formulated as such: "What are the quantities that the n industries of the economy need to produce in order to meet a *given* final demand?" (Smith, 1951). Such an approach can cover a broader variety of economic situations. For example, it can analyze an open economy in which part of the final demand consists in foreign demand.
Let us denote with $\mathbf{A}$ the matrix containing only the first n rows and columns of matrix $\tilde{\mathbf{A}}$, the ones containing the inter-industrial coefficients. Let $\mathbf{y}$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{3.16}$$

be the vector of given final demand, and, as before, $\mathbf{q}$ the vector of the quantities produced. Denoting, as in section 3.1, with L_i the amount of labor employed by industry i let

$$l_i = \frac{L_i}{Qi} \tag{3.17}$$

be the amount of labor necessary to produce one unit of commodity i, $\mathbf{l}$ the vector containing all such coefficients and the scalar L as the quantity employed in the system. We can then write the following system of equations in matrix form

$$\begin{cases} \mathbf{Aq} + \mathbf{y} = \mathbf{q} \\ \mathbf{l}'\mathbf{q} = \mathbf{L} \end{cases} \tag{3.18}$$

Anyway, since we consider $\mathbf{y}$ as given, it is better to re-write the systems above in the following way

$$\begin{cases} (\mathbf{I} - \mathbf{A})\mathbf{q} = \mathbf{y} \\ \mathbf{l}'\mathbf{q} = \mathbf{L} \end{cases} \tag{3.19}$$

or, in extended form

$$\begin{bmatrix} 1 - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & 1 - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & 1 - a_{nn} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{3.20}$$

and

$$l_1 Q_1 + \cdots + l_n Q_n = L \tag{3.21}$$

Notice that it would be theoretically possible to construct an analogous system for the determination of prices, of the form

$$\mathbf{p}'(\mathbf{I} - \mathbf{A}) = \mathbf{v} \tag{3.22}$$

Where $\mathbf{v}$ would be a given vector of value added. Such a formulation, anyway, did not receive much success, also because it would presume an exogenous determination of the values of incomes in the economy, through what Leontief called "value-added equation". This simplistic separation between the distributive process and the determination of prices is indeed very problematic, for reason that we will investigate in greater depth in Chapter 4 (see section 4.3.1), as pointed out byKurz and Salvadori (2006). In general, when the "Leontief Open Model" is mentioned, people refers to the system for quantities 3.20.

Let us now see the solution of the system.

Proposition 3.4.1. *Consider a matrix of technical coefficients* $\boldsymbol{A}$*. If the greatest eigenvalue of matrix* $\boldsymbol{A}$ *is smaller than one*[8]*, then system 3.19 has a unique, non-negative solution*

$$\mathbf{q} \geq \mathbf{0}, \quad L > 0 \tag{3.23}$$

Proof. The existence of the solution is given by the fact that, because of the same reasoning of section 3.3, we can assume the matrix $(\mathbf{I} - \mathbf{A})$ to be composed by n linear columns. Therefore it has full rank and consequently an inverse. The solution is then uniquely determined in

$$\mathbf{q} = (\mathbf{I} - \mathbf{A})^{-\mathbf{1}}\mathbf{y} \tag{3.24}$$

In order to show that the solution $\mathbf{q}$ is non negative we make use of the hypothesis regarding the largest eigenvalue of $\mathbf{A}$. Since $\mathbf{y}$ is non negative if we could prove that the matrix $(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}$ is non-negative we would have proven the non-negativity of $\mathbf{q}$. We can prove this making use of Proposition 10.3.5 of Mathematical Appendix, considering $\mu = 1$ and noticing that the hypothesis tells us exactly that $\mu > \lambda_{max}$. So we have

$$(\mathbf{I} - \mathbf{A})^{-\mathbf{1}} \geq \mathbf{0} \tag{3.25}$$

and the demonstration is complete. □

Let us consider some consequences arising from proposition 3.4.1.
First of all, the quantity of labor employed, L, is now a solution and not any more, as in the closed model, the total amount of labor available. So now it is a variable and we have to consider that $L \leq Q_{n+1}$.
It is worth also to spend some time in the interpretation of the coefficients of the inverse matrix $(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}$. Consider an economy described by

$$(\mathbf{I} - \mathbf{A})\mathbf{q} = \mathbf{y} \tag{3.26}$$

Let us now make the (mental) experiment of varying the amount of commodities produced in the various industries in such a way that the net product of industry j is equal to one, while the net product of all other industries is equal to zero. This consists in finding the column vector $\alpha_{\mathbf{j}}$ such that

$$(\mathbf{I} - \mathbf{A}) \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{nj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \longleftarrow \text{j-th position} \tag{3.27}$$

or

$$(\mathbf{I} - \mathbf{A})\alpha_{\mathbf{j}} = \mathbf{e_j} \tag{3.28}$$

[8]The economic meaning of this apparently arbitrary mathematical condition will be investigated in section 4.3.4. For now, it is enough to say that it is a condition that must be satisfied by any economy worth of study.

This operation can always be made if the matrix of technical coefficients satisfies the hypothesis of Proposition 3.4.1, since it consists in finding the solution for the system 3.26 when $\mathbf{y}$ is equal to the selection vector $\mathbf{e_j}$.
Compare expressions 3.26 and 3.28. In 3.28 vector $\alpha_\mathbf{j}$ takes the place of $\mathbf{q}$ and vector $\mathbf{e_j}$ takes the place of $\mathbf{y}$. This means that when the economy operates at level $\mathbf{q}$ it is necessary to produce Q_1 unit of commodity one, ..., Q_n unit of commodity n in order to obtain, as net product, y_1 units of net product of commodity 1, y_2 units of net product of commodity 2 etc. At the same time, when the economy operates at level $\alpha_\mathbf{j}$, it is necessary to produce α_{1j} units of commodity 1, α_{2j} units of commodity 2, ..., α_{nj} units of commodity n to obtain *just one unit* of net product of commodity j. Therefore the vector $\alpha_\mathbf{j}$ contains the requirements from all the sector of production necessary to produce one unit of net product of commodity j. Since the matrix **(I-A)** is invertible, we have

$$\alpha_\mathbf{j} = (\mathbf{I} - \mathbf{A})^{-\mathbf{1}}\mathbf{e_j} \tag{3.29}$$

Which means that the $j - th$ column of matrix $(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}$ represents the quantities of commodities that have to be produced by the whole system in order to obtain one unit of net product of commodity j.
Repeating the above procedure for all commodities, we obtain that

$$(\mathbf{I} - \mathbf{A})^{-\mathbf{1}} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix} \tag{3.30}$$

Therefore we can call matrix $(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}$ the *matrix of total requirements*, as opposed to the matrix of technical coefficients that represent the *matrix of direct requirements*. If element a_{ij} represented the amount of i directly required to produce an unit of j, an element of $(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}$, α_{ij}, represent the amount of i that must be produced overall to obtain one unit of net product of j. This matrix has become known as *Leontief's Inverse*.

3.5 Conclusions

As we have said, Leontief's work was seminal in the recovery of a vision of the economic system as a circular flow, in which commodities are produced by means of other commodities. This point of view is presented initially through the description of the inter-industrial relationships of the economy in the Input-Output Table and later on formalized in two theoretical models, which rely upon specific assumptions regarding the technology.
It is worth to point out some particular features underlined by Leontief's models.

First of all, the singular duality between the determination of equilibrium prices and equilibrium quantity. Such characteristic will be underlined also by Neumann (1945), as

we will see in Chapter 5. Notice that in this context "equilibrium" is intended as "necessary for the reproduction of the system", not, in neoclassical terms, as the result of the interaction between supply and demand. This allows us to make an other consideration on the characteristic of the presented models: in Leontief's Closed Model, the prices depend uniquely from the technical coefficients of the economy, as well as necessaries quantities. Demand and supply do not play any role in this process (Kurz and Salvadori, 2006).

Regarding the Open Model, the Leontief's Inverse constitutes a powerful theoretical tool to analyse in which way the creation of a surplus for a single commodity depends on the complex inter-correlation between all the industries in the economy. On the other hand, we have already mentioned the limits deriving from the removal of the issue of distribution from price determination. The approach proposed by Sraffa, which was working at his model in the same period of Leontief, deals with this problematic issue in greater depth.

Chapter 4

Production of Commodities by Means of Commodities

4.1 Introduction

In the same period in which Leontief was developing his input-output model, an other young scholar was independently (Samuelson, 1991) developing a rather similar approach to the study of economic phenomena, which was also based on the classical vision of the economy as a circular flow.

Piero Sraffa was born in Turin in the year 1989. He graduated in 1920 with a dissertation with the title *"L'inflazione monetaria in Italia durante e dopo la guerra"* ,[1] supervised by Luigi Einaudi. He lived in Italy until 1927 when, because of the difficulties related to his anti-fascist ideas, he moved in England, where he will reside until his death, in 1983.
Despite his intellectual production consisted in only few articles, a critic edition of Ricardo's works, and a 100-pages book, his works has been very influential and it is by many scholars considered a fundamental column in the history of economic thought. He has been intellectually and personally very close to three of the main cultural protagonists of its time (Roncaglia, 2006) : Antonio Gramsci, John Maynard Keynes, and Ludwig Wittgenstein.

Having met Gramsci at the University where he was studying, it was him that made sure that the co-founder of the Italian Communist Party received books and magazines during his ten-years imprisonment, and remained in contact with him until its death.

Sraffa attracted the attention of Edgeworth, and consequently of Keynes who was with him the co-director of the "Economic Journal", thanks to an article published in 1925, *Sulle relazioni tra costo e quantità prodotta*,[2]in which he strongly criticizes Marshall's theory of production, in particular the theoretical foundations of its assumptions

[1]Monetary inflation in Italy before and after the war.
[2]On the relationship between cost and quantity produced.

of the "laws of return". It is worth to spend some words on the critique he poses to the law of diminishing returns in particular, since it is related with the subject of this dissertation. Sraffa (1925) states that such law, to be effective, must depend on the employment of a scarce factor of production. But the increase of average cost resulting from the increase in production of the industry in consideration (unless we consider as an industry the set of firms employing that factor of production) must result in a proportional increase in the average cost of all other industries employing the same factor of production, killing the *ceteris paribus* assumption, fundamental for Marshallian analysis. This is a first consideration on the complex interrelationships that characterize the determination of prices in an economy considered as a circular flow, where commodities produces other commodities.

As we were saying, Sraffa was noticed by Keynes, who offered him a place as *lecturer* at the University of Cambridge and the possibility of publish his article on the "Economic Journal". Sraffa actually substantially modified its article, which was published in 1926 with the title of "The Laws of Returns under Competitive Conditions" (Sraffa, 1926), adding an original part that would set the basis for the study of Imperfect Competition.[3] In Cambridge he met and become close to Wittgenstein, on whose though he will have a substantial influence: Sraffa is specifically cited in the preface of Witgennstein's second work, *Philosophical Investigations*.[4]

Anyway, Sraffa's most important work is surely "*Production of commodities by means of commodities*" (Sraffa, 1960), which he elaborated during a period which lasted more than 30 years. The subheading of the book is " Prelude to a Critique of Economic Theory", from which is clear that one of the main aim of Sraffa was to move away from the marginalistic method, in favour of a return to the approach used by the classical economists. The main features that characterize this approach are: the inquiry on an economy based on the division of labor; the production of a surplus, to be divided among social classes; the circular nature of the production process.
Saying that in Sraffa's model we have division of labor it means that each firm's production is not enough to satisfy its need for means of production, and therefore is forced to trade with other firms in order to obtain them. The fact that each commodity can be both an input and an output (since commodities are produced by means of other commodities) underline the circularity of the production process, and mark a clear distinction of (at least a big part of) neoclassical tradition, which, in Sraffa's words, sees the economy as a "one way avenue that leads from 'Factors of Production' to 'Consumption Goods' ". The main topic of inquiry of Sraffa's dense book consist in what in the classic theory is denoted as "the problem of *value*", that is the interrelation between the necessary exchange that have to take places between industries in order to recover the

[3]Sraffa was the first one to suppose downward sloping demand curves for the individual firms (Roncaglia, 2006).

[4]Wittgensten writes: " Even more than to this [..] criticism I am indebted to that which a teacher of this university, Mr. P. Sraffa, for many years unceasingly practised on my thoughts. I am indebted to this stimulus for the most consequential ideas of this book."

necessary means for the production, and the distribution of the surplus (potentially) generated by the economy among the different social classes.

In the following sections we will present (part of) Sraffa's work in "Production of commodities", differentiating between an economy that produces just the strictly necessary for reproduction (section 4.2) and an economy which is able to produce a surplus (section 4.3)

4.2 Production for Subsistence

Sraffa begins his book considering a very simple economy, composed by single product industries using only circulating capital, which produce just enough to reproduce itself (Sraffa, 1960). This means that the total amount of a commodity used in all sectors of production is *exactly equal* to the amount of the same commodity produced by the corresponding industry.

Before going into rigorous formulation, let us consider the example made by Sraffa himself in the firs chapter of *Production of Commodities*: a primitive, agricultural economy producing only two goods - wheat for the subsistence of those who work and iron for the construction of agricultural tools. Sraffa assumes that, in order to produce 400 quarters of wheat, the wheat industry uses 280 quarters of wheat and 12 tons of iron; at the same time, the iron industry uses 120 quarters of wheat and 12 tons of iron to produce 20 tons of iron. The conditions of production can be summarized as follow:

$$\begin{aligned} 280q(wheat) + 12t(iron) &\longrightarrow 400q(wheat) \\ 120q(wheat) + 8t(iron) &\longrightarrow 20t(iron) \end{aligned} \tag{4.1}$$

It is immediate to see that the system is in a self-replacing state, meaning that is able to produce at least the commodity it has employed: 400 quarters of wheat are produced, and 400 $(280 + 120)$ quarters are used; analogously 20 $(8 + 12)$ tons of iron are used as means of production and 20 tons are produced.

It is also clear that there is an *unique set* of exchange value that allow the system to *reproduce itself*: indeed the wheat industry, which at the beginning of the production process was in possess of all the means of production necessary to activate it, now holds only its specific kind of commodity. Therefore it needs to obtain in the market 12 tons of iron, and can offer in exchange 120 quarters of wheat (since 280 quarters are needed as means of production). At the same time, not surprisingly (given that the economy is build on the assumption of production for subsistence only) the iron industry needs 120 quarters of wheat and can exchange on the market 12 tons of iron. The value of 120 quarters of wheat will therefore be equal to the value of 12 tons of iron, or, identically, one ton of iron will be worth 12 quarters of iron.

These relative prices would be obtained as the solutions of the following system, where the price of iron is set as *numeraire*

$$\begin{cases} 280p_g + 12p_i = 400p_g \\ 120p_g + 8p_i = 20p_i \\ p_i = 1 \end{cases} \tag{4.2}$$

We can make a first observation out of this very simple example: prices ultimately depend on quantity produced, which in turn depend on given methods of production. Therefore we can say that prices are actually determined by methods of production (Finoia, 1979), a notion that will be fundamental also in more advanced representations of the economy. Let us now formulate a more rigorous and general representation. Consider an economy in self-replacing state with n single-product industries using only circulating capital. Recall the definition 3.4 of production coefficients: a production coefficient correspond to the amount of commodity i necessary for the production of *one unit* of commodity j by the correspondent industry. We call have called $\mathbf{A}$ the *Matrix of technical coefficients*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{4.3}$$

The columns of $\mathbf{A}$ represent the industries in an economy: the elements in the j-th column are the quantities of each commodity in the system used by industry j to produce one unit of its proper commodity.

The rows of $\mathbf{A}$, on the other hand, refers to the single commodities: the elements of the i-th row are the amount of commodity i used in the production of all other commodities in the system.

Of course we can easily express our simple example of a two-goods economy in this form (Pasinetti, 1986). Calling $\mathbf{q}$ the quantity vector, the conditions of production of the above economy can be represented as:

$$\mathbf{Aq}=\mathbf{Iq} \tag{4.4}$$

$$\underbrace{\begin{bmatrix} \frac{280}{400} & \frac{120}{20} \\ \frac{12}{400} & \frac{8}{20} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 400 \\ 20 \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}} \underbrace{\begin{bmatrix} 400 \\ 20 \end{bmatrix}}_{\mathbf{q}} \tag{4.5}$$

Note that it is in general convenient (Newman, 1962) to choose the unit of measure of the commodities in a way such that the quantity produced is, for every commodity,

equal to one.[5] In this way, the technical coefficient a_{ij} represent the *proportion* of the commodity i used in the production of commodity j.

Given this convention, that we will implicitly use from now on, our simple two goods economy would look like

$$\mathbf{Aq}=\mathbf{Iq} \tag{4.6}$$

$$\underbrace{\begin{bmatrix} \frac{280}{400} & \frac{120}{400} \\ \frac{12}{20} & \frac{8}{20} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{q}} \tag{4.7}$$

Note that the matrix $\mathbf{A}$ has changed with respect of 4.5. This is not surprising, since the technical coefficients are expressed in *unit of measure commodity i / unit of measure commodity j*, and clearly their value will change changing the unit of measure.

Notice that this representation is similar to the one we have seen in section 3.3, the Leontief Closed Model. Nevertheless an important difference must be considered: Leontief made an explicit assumption of constant return to scale, while Sraffa made an explicit statement of *non assuming* constant return to scale (in the Preface of Sraffa (1960)). While the matrix coefficient in Leontief model is assumed to be constant for any scale of production (which permitted the construction of the coefficient matrix itself by dividing the observed quantities of inputs by the observed quantities of output, and permitted to treat the level of production as variable), in Sraffa nothing can be said about technical relationship at other scales of production. The quantities are given, and, with the appropriate choice of an unit of measure, those are the quantities of inputs necessaries to produce one unit of each commodity. System 4.7 therefore can be considered as a "snapshot" of an economy in a particular moment, and cannot be considered as a theory regarding the general conditions of production of an economy.

Let us now define some properties of the matrix of technical coefficients $\mathbf{A}$. Let $\mathbf{I}$ be the identity matrix of order n and $\mathbf{s}$ the (row) sum vector. Then:

Property I. The matrix $\mathbf{A}$ is non-negative.

$$\mathbf{A} \geq \mathbf{0} \tag{4.8}$$

This is clear since a negative technical coefficient would not have economic meaning. On the other hand, nothing says that any commodity must enter in the production process

[5] In our example, say, the unit of measure of wheat become (400tons), which we may call T. Therefore 400 tons become 1 T, 800 tons become 2T, 200 tons become $\frac{1}{2}$T, 280 tons become $\frac{280}{400}$T and so on.

of all other commodities. Therefore some elements of **A** can be zeros

Property II. Every column contain at least a strictly positive element.
This means that

$$\mathbf{sA} > \mathbf{0} \tag{4.9}$$

This because any commodity needs at least one commodity to be produced.

Property III. The sum of the coefficients of all the rows of **A** is equal to 1.

$$\begin{cases} a_{11} + a_{12} + \cdots + a_{1n} = 1 \\ a_{21} + a_{22} + \cdots + a_{2n} = 1 \\ \vdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} = 1 \end{cases} \tag{4.10}$$

or

$$\mathbf{(I\ -\ A)s{=}0} \tag{4.11}$$

This is the formal requirement for the assumption of self-replacement for subsistence. In fact, it means that, for each commodity, the amount used in all industry as mean of production is equal to the quantity of that commodity that has been produced this year (that, remember from above, it has been conventionally set equal to one).
This property has two important consequences:

Consideration I. Matrix **I-A** has not full rank. Actually

$$rank[\mathbf{I\text{-}A}] = n - 1 \tag{4.12}$$

In fact, given Property 3, each column of **I-A** can be expressed as the sum of the other columns (Marangoni, 1985), therefore **I-A** cannot have full rank. At the same time, there is no economic reason for which the other columns would be linearly dependent, so we will assume they are not.

Consideration II The matrix **A** is *irreducible* (see Mathematical Appendix).
This condition can be read as the requirement that no independent sub-economy exist. A subgroup S of industries is an *independent sub-economy* (Afriat, 2006) if

$$a_{ij} = 0 \quad \text{for } i \in S,\, j \in \bar{S} \tag{4.13}$$

That is industries belonging to S do not use for their production processes any commodity produced by industries that do not belong to S. An analogous way of posing this condition

is saying that commodities must be *interdependent*. A commodity j is said to *depend* on commodity i if $a_{ij} > 0$. A group of commodities are *independent* if each one of them is independent of all other commodities outside the group. Thus if commodities $1, \ldots, r$ are independent the technical coefficient matrix $\mathbf{A}$ can be rearranged in the following form

$$\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{0} & \mathbf{A_{22}} \end{bmatrix} \tag{4.14}$$

where A_{11} is a (r, r) matrix. Commodities are interdependent if no independent proper subgroup exist. The interdependence property is automatically satisfied in an economy for subsistence, where "rank equally, each of them being found both among the products and among the means of production" (Sraffa, 1960), and so each commodity, directly or indirectly, enter in the production of all other commodities. The economic interpretation of this characteristic of matrix $\mathbf{A}$ is a quite important one, but will be analysed in greater depth in section 4.3.

Notice that while Property I and II are typical of the matrix of coefficients of any meaningful economy, Property II is valid only in the case of a self-replacing economy for subsistence only and will be dropped later on.

Let us now turn our attention at the determination of prices. As in our simple two-commodity example, at the end of the production period a certain kind of a commodity is exclusively in the hands of the industry which has produced it. In order for the production process to start over the next period, we have to find a set of exchange value for which, in the market, each industry is able to obtain the same amount of commodities which was in its possession at the beginning of the period. We have, therefore, to find the *price vector* $\mathbf{p}$

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \tag{4.15}$$

for which the value of each production is exactly equal to the value of the means of production used, for each industry. This means solving the system

$$\begin{cases} a_{11}p_1 + a_{21}p_2 + \cdots + a_{n1}p_n = p_1 \\ a_{12}p_1 + a_{22}p_2 + \cdots + a_{n2}p_n = p_2 \\ \vdots \\ a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{nn}p_n = p_n \end{cases} \tag{4.16}$$

or, in synthetic form

$$\mathbf{p'(I\text{-}A)=0} \tag{4.17}$$

Let us now prove that in a self-replacing economy for subsistence in which the above properties hold, there always exists a non-negative vector of prices which is a solution of

the system 4.17.

Proposition 4.2.1. *Let $\boldsymbol{A}$ be a matrix satisfying Properties I and II.*[6]
Let λ_{max} denote the highest eigenvalue of matrix $\boldsymbol{A}$. Then

$$\lambda_{max} > 0 \tag{4.18}$$

Proof. Since $\mathbf{A} \geq \mathbf{0}$ Proposition 10.2.4 (Mathematical Appendix) assures that $\lambda_{max} \geq 0$. Proposition 10.2.9(Mathematical Appendix) says that

$$\lambda_{max} \geq \min_i \mathbf{a_i s} \tag{4.19}$$

where a_i is a column of $\mathbf{A}$. Then, because of Property II of the matrix of technical coefficients,

$$\lambda_{max} > 0$$

□

Proposition 4.2.2. *Let $\boldsymbol{A}$ be a matrix satisfying Properties I, II and III.*[7] *The the highest eigenvalue of $\boldsymbol{A}$ is then equal to one.*

$$\lambda_{max} = 1 \tag{4.20}$$

Proof. Let $\mathbf{x}$ be the left eigenvector associated with λ_{max}. Note that by Proposition 10.2.3 (Mathematical Appendix) we have that $x > 0$.
By definition

$$\mathbf{x'A} = \lambda_{\mathbf{max}}\mathbf{x'} \tag{4.21}$$

Right multiplying both members for $\mathbf{s}$ we obtain

$$\mathbf{x'As} = \lambda_{\mathbf{max}}\mathbf{x's} \tag{4.22}$$

At the same time, left-multiplying expression 4.11 (of Property III) for **x'** we have

$$\mathbf{x's} = \mathbf{x'As} \tag{4.23}$$

By the two previous expressions we therefore have that

$$\mathbf{x's} = \lambda_{\mathbf{max}}\mathbf{x's} \tag{4.24}$$

from which, given that $x > 0$

$$\lambda_{max} = 1 \tag{4.25}$$

□

[6] non negative matrix in which each commodity needs at leat one commodity to be produced
[7] self-replacing economy for subsistence.

Proposition 4.2.3. *Let **A** be a matrix satisfying Properties I, II and III. Then there exist at least a non negative price vector **p** which is the solution for system 4.17.*

$$\exists \mathbf{p} > \mathbf{0} \quad \textit{s.t: } \boldsymbol{p'(I\text{-}A)=0} \tag{4.26}$$

Proof. Property III implies that

$$Det[\mathbf{I\text{-}A}] = 0 \tag{4.27}$$

that is a necessary and sufficient condition for the system to have solution different that the obvious one $\mathbf{p}$=$\mathbf{0}$.
Given Proposition 4.2.2 a vector $\mathbf{p}$ satisfying system 4.17 satisfies also

$$\mathbf{p'A}=\lambda_{max}\mathbf{p'} \tag{4.28}$$

since

$$\lambda_{max} = 1$$

The above condition shows that $\mathbf{p}$ is the left eigenvalue of the non-negative matrix $\mathbf{A}$. Therefore Proposition 10.3.2(Mathematical Appendix) for non-negative, irreducible matrices guarantees that it is positive. □

Notice that Proposition 4.2.3 do not guarantee the uniqueness of vector $\mathbf{p}$. Actually, because of Property III, the systems contains n variables but only $n-1$ independent equations. The problem can be solved by setting the price of one of the commodities equal to one.

$$p_i = 1 \quad \text{for some} \quad i = 1, ..., n \tag{4.29}$$

In this case the system is uniquely determined.

In this way we have proved that in an economy in self-replacing state for subsistence only it is always possible to find a set of *relative* prices which are all strictly positive (and therefore, economically meaningful), and that permits the restoration of the original condition for the production.

It is worth to spend some words on the *economic meaning* of the prices that we have just determined: those are the *necessary* prices for the economy to reproduce itself. The idea behind this approach to value is that an economy must satisfy certain requirements in order to be able to reproduce itself. Some are physical requirements, meaning that an economy must produce at least the amount of commodities that it has consumed in the production process (in the case we are considering, the exact amount).[8]But others are *economic* requirements: if the exchange value resulting from the market would be different from the ones determined by system 4.17, then some firms would incur in a

[8]Notice that this requirement means that the the vector of the total productions of industries must be the only solution of the system 3.9 of Leontief Closed Model for the matrix of technical coefficients determined from the observed quantities.

loss, while other would generate a profit. But the industries incurring in a loss would not be able to recover their means of production, and therefore would stop producing, but this would make also some of the apparently more fortunate firm unable to recover their means of production (for a very simple reason: they are not produced any more) (Gilibert, 2006).
Sraffa's prices are therefore the only possible firm for which a viable economy can survive. The fundamental point is that, if we consider an *actually existing* economy in self replacing state, well then, for the same fact that it exist, it means that the resulting prices *must be* those resulting from system 4.17.

Let us now consider how this considerations evolve if the economy is capable of producing more than it need for strict reproduction, i.e. if it is able of generating a *surplus*.

4.3 Production of a Surplus

If the economy is capable of producing more than it is strictly necessary for the renovation of its means of production, a *surplus* arises.
We can see an example of an economy with a surplus simply slightly modifying our two-goods economy analysed in the previous section. Following Sraffa (1960) let us assume that the wheat industry is able, using the same inputs of before, to produce 575 quarters of wheat instead of 400. In this case system 4.1 become:

$$\begin{aligned} 280q(wheat) + 12t(iron) &\longrightarrow 575q(wheat) \\ 120q(wheat) + 8t(iron) &\longrightarrow 20t(iron) \end{aligned} \tag{4.30}$$

The surplus for this economy consists in 175 quarters of wheat, and derives from an improved *method of production* in the wheat industry.
If we want to represent system 4.30 in a form analogous of that one in 4.7 we have to add an other element, the *surplus vector* $\mathbf{y}$, in which the element y_i represent the (physical) quantity of commodity i that exceed the amount used as mean of production in all the industries in the economy. We can then write (remembering that we assume units of measure for commodities such that $\mathbf{q}$ is composed only by ones)

$$\underbrace{\begin{bmatrix} \frac{280}{575} & \frac{120}{575} \\ \frac{12}{20} & \frac{8}{20} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{q}} + \underbrace{\begin{bmatrix} \frac{175}{575} \\ 0 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{q}}$$

or

$$\mathbf{Aq} + \mathbf{y} = \mathbf{Iq} \tag{4.31}$$

Notice that this is a system similar to the one we have found in section 3.4, the Leontief Open Model, and the same considerations made in previous section with respect to the similarities with Closed Leontief Model are of course still valid.

We have now to reconsider the properties of matrix **A** that we have introduced in the previous section. As mentioned before, Property I and II are characteristic of any meaningful economy. Property III, on the other hand, must be dropped. We substitute it with the following:

Property III.bis. The sum of the coefficients of all the rows of **A** is strictly smaller than 1.[9]

$$\begin{cases} a_{11} + a_{12} + \cdots + a_{1n} < 1 \\ a_{21} + a_{22} + \cdots + a_{2n} < 1 \\ \vdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} < 1 \end{cases} \tag{4.32}$$

or

$$\mathbf{(I - A)s > 0} \tag{4.33}$$

This means that all industries produce an amount of commodity greater than the amount used as mean of production during one year. An economy that satisfied this condition is said to be *viable*.

What about then Consideration I and II, that originated from Property III?

Consideration I has no more reason to be true. Indeed, since methods of production have generally different structures, there is no economic reason to think that the columns of **A** are linearly dependent. In general, therefore, it will hold:

$$rank[\mathbf{I\text{-}A}] = n \tag{4.34}$$

Also Consideration II has no reason to hold any more.
Actually, the emergence of a surplus allow the existence of a new kind of commodity that had no place in the Production for Subsistence economy, a commodity that it is not used in the production of any other commodity. Sraffa defines: *basic commodity* a commodity that, directly or indirectly, is used in the production of *all* other commodities; *non-basic commodity* a commodity that it is not a basic commodity. Notice that it is important to notice that a commodity may enter indirectly in the production of other commodities. For example, wheat may not be used directly for the production of cars (say that cars are produced by iron, rubber and plastic), but if it enters in the production process of iron, it consequently enters indirectly also in cars' one. Consider the following example of an economy that produces two basic commodities and a non-basic one, taken from Finoia (1979).

[9] Actually Sraffa define self-replacing state with surplus as a system in which this is true for *at least one* row of **A**. I follow Marangoni (1985) in this slight modification.

$$
\begin{aligned}
250q(wheat) + 10t(iron) &\longrightarrow 400q(wheat) \\
120q(wheat) + 8t(iron) &\longrightarrow 20t(iron) \\
500q(wheat) + 2t(iron) &\longrightarrow 100l(whisky)
\end{aligned}
\tag{4.35}
$$

The surplus of the system is therefore 5 tons of iron and 100 litres of whisky. As we can easily see, whisky is not used in the production of any other commodity (not even in its own production), and therefore is considered a non-basic commodity.
Now, the existence of non-basic goods has important mathematical consequences. In fact, if a non-basic commodity exists, it means that the matrix $\mathbf{A}$ is *reducible*. This means that it is possible, by appropriate permutations of $\mathbf{A}$, rewrite the matrix of technical coefficients in such a way that the first r rows of the matrix represent basic commodities, while the last $n-r$ rows are non-basic commodities. The matrix would then assume the form

$$
\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{0} & \mathbf{A_{22}} \end{bmatrix} \tag{4.36}
$$

where :
$\mathbf{A_{11}}$ is an irreducible (r,r) matrix, since it represent the use of basic commodities in basic-commodities industry.
$\mathbf{0}$ is a null (n-r, r) matrix. The fact that $\mathbf{0}$ is null is clear if we recall that the columns of $\mathbf{A}$ represent the industries of the economy. Therefore, by definition, all the coefficients relative to the non-basic commodities do not enter in the production of the basic commodities.
$\mathbf{A_{22}}$, on the other hand, contain the use of non-basic commodities by non-basic commodities industries, and can be irreducible or reducible. If it is irreducible, we have to assume that at least an element of $\mathbf{A_{12}}$ must be strictly positive, otherwise $\mathbf{A_{11}}$ and $\mathbf{A_{22}}$ would represent distinct, unconnected economic systems.

The difference between basic and non-basic commodities is fundamental in determining the role of a commodity in the determination of prices and distribution. We will therefore come back on this distinction later on.

The distributive variables

Let us now consider the distributive consequences arising from the existence of a surplus. In the Production for Subsistence model of the previous section the distribution problem was a matter of technology: every industry needed a certain amount of commodities, and the distribution process, via the prices determined by system 4.17, made sure that every industry would receive exactly what it needed. But with the emergence of a surplus, once every industry has restored its means of production, there is an amount of commodities unclaimed. This creates a problem in the determination of a system of prices analogous

of 4.17. Analytically, the rank of $\mathbf{A}$ is now equal to n while the number of variables remain $n-1$, so that the system is undetermined. One could think to solve this problem by distributing the surplus *before* the price are determined, but, as we will see in the following paragraph, this is not possible.

Sraffa assumes that *Net Product* of the economy (the exchange value of the commodities in the surplus) is distributed to the members of the communities in two forms: profits (to the owners of the means of production) and salaries (to the workers). Sraffa makes some important assumptions regarding these two distributive variables.

The rate of profit π is considered to be uniform among all industries. This means that the value of a unit of circulating capital in any production process is remunerated in the same way.[10] This creates a problem with the possibility mentioned above of distributing the surplus before the determination of prices. If profits have to be distributed in proportion to the means of production employed this means that different means of production employed in different industries must me compared. But such an heterogeneous agglomerate of commodities cannot be compared in any way without a price system. On the other hand we cannot even determine the value of π *after* we have determined the price system, since it is clear (but we will see it more clearly in the next subsection) that prices depend on the rate profit. This means that the distribution of the surplus must happen *at the same time* and *through the same mechanism* of the determination of prices (Sraffa, 1960).

The unit salary w is, on the other hand, distributed proportionally to the physical quantity of labour employed. It is assumed that in the system the quality of labour is homogeneous. It is worth noticing here a difference between the conception of salary used by Sraffa and the tradition of Classical Economy, since for many other aspects Sraffa's approach is explicitly inspired by that tradition. Classical economist, in fact, used to assume that the salary were at the subsistence level, being the value of the commodities that would permit the worker to survive and go to work an other day. On the other hand, Sraffa investigate the effect of the possibility that workers claim part of the net product. As Sraffa (1960) points out it would be methodologically more correct to distinguish between the two "natures" of the salary and keep the "subsistence part" inside the means of production (economically analogous with the gasoline necessary to start an engine) and consider as a distributive variable only the "surplus part", but does not follow this path to not make things too confused.[11] Moreover, the salary is assumed to be distributed entirely at the end of the production process, as a portion of annual product (Sraffa, 1960). Some have seen in this departure from the tradition (that usually considered

[10] There have been several interpretation on the economic meaning of this assumption, but we will not discuss them here. It is in general a traditional assumption of classical economists to assume constant rate of return in all sectors of production.

[11] "An inconvenient of this course is that it involves relegating necessaries of production in the limbo of non-basic products", (Sraffa, 1960) pag 12. Considering the different role that basin and non-basic commodities play in the determination of prices, it would be opportune to find alternative way through which the method of production of subsistence goods affect the system, like imposing a lower bound to salaries.

an advanced wage), besides the analytical reason of simplification of the system, also an underlying of the difference between *produced* means of production and *unproduced* labor in the determination of the value of production (Bortis, 2013).

Our matrix of technical coefficients needs then to be rearranged, as the following example shows.[12]
Consider the following economic systems with three goods

$$\begin{aligned} 224q(wheat) + 40t(iron) + 80t(coal) &\longrightarrow 480q(wheat) \\ 69q(wheat) + 90t(iron) + 135t(coal) &\longrightarrow 180t(iron) \\ 165q(wheat) + 50t(iron) + 150t(coal) &\longrightarrow 450t(coal) \end{aligned} \tag{4.37}$$

Consider then the annual subsistence of a worker is 0,05 tons of coal and 0.03 quarters of wheat. Assume then that the wheat industry employs 800 workers, the coal industry 500, and the iron industry 300. Then we can rewrite the system above (subtracting from each quantity employed the necessary in order to pay the workers, and including the workers themselves) amount as

$$\begin{aligned} 200q(wheat) + 40t(iron) + 40t(coal) + 800(workers) &\longrightarrow 480q(wheat) \\ 60q(wheat) + 90t(iron) + 120t(coal) + 300(workers) &\longrightarrow 180t(iron) \\ 150q(wheat) + 50t(iron) + 125t(coal) + 500(workers) &\longrightarrow 450t(coal) \end{aligned} \tag{4.38}$$

Notice that the net product in the second system is higher, but this is clear since the means of subsistence of the workers are extracted by the means of production.

4.3.1 The system of prices

For reason of convenience, let us assume that the quantity of labour is measured in such a way that the total quantity employed is equal to one. Defining

$$\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ ln \end{bmatrix} \tag{4.39}$$

as the *vector of labor coefficients*, where the single element l_i represent the amount of labor employed annually in the $i-th$ industry, we then have

$$\sum_{i=1}^{n} l_i = 1 \quad or \quad \mathbf{l's} = \mathbf{1} \tag{4.40}$$

Moreover we will assume that a certain amount of labour is necessary for the production of any commodity, so that no completely automatic production exists. This is equivalent to say that the vector of labor coefficients is strictly positive, or

$$\mathbf{l} > \mathbf{0} \tag{4.41}$$

[12]Notice that this rearranging does not influence Property I, II and IIIbis

Determined that, and given and the assumption of the previous subsection regarding the distributive variables, the vector of prices $\mathbf{p}$ and the two distributive variables π and w will be determined by the following system:

$$\begin{cases} (a_{11}p_1 + a_{21}p_2 + \cdots + a_{n1}p_n)(1+\pi) + l_1 w = p_1 \\ (a_{12}p_1 + a_{22}p_2 + \cdots + a_{n2}p_n)(1+\pi) + l_2 w = p_2 \\ \vdots \\ (a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{nn}p_n)(1+\pi) + l_2 w = p_n \end{cases} \tag{4.42}$$

which can be rewritten as

$$\mathbf{p}'\mathbf{A}(1+\pi) + \mathbf{l}w = \mathbf{p}' \tag{4.43}$$

or

$$\mathbf{p}'[\mathbf{I} - \mathbf{A}(1+\pi)] = \mathbf{l}w \tag{4.44}$$

The rate of profits, uniform for all industries, represent the ration between the value of the part of the Net Product going to the owners of means of production and the value of circulating capital employed. Let ω be the fraction of Net Product going to workers. Then

$$\pi = (1-\omega)\frac{\mathbf{p}'(\mathbf{I}-\mathbf{A})\mathbf{s}}{\mathbf{p}'\mathbf{A}\mathbf{s}} \tag{4.45}$$

To avoid the arbitrary choice of a commodity as *numeraire*, Sraffa chooses the convention of setting the Net Product of the system, intended as a composite commodity, equal to one. This means

$$p_1[1 - (a_{11} + a_{12} + ... + a_{1n})] + ... + p_n[1 - (a_{n1} + a_{n2} + ... + a_{nn})] = 1 \tag{4.46}$$

or

$$\mathbf{p'(I\text{-}A)}\mathrm{s} = 1 \tag{4.47}$$

Notice that actually, given the conventions 4.47 and 4.40, the unit salary w is equal to the proportion of total Net Product going to workers. Indeed, given that Net Product is divided between salaries and profits, total salaries (denoted W) are the proportion ω of Net Product going to workers.

$$W = \omega\mathbf{p'(I\text{-}A)}\mathrm{s} \tag{4.48}$$

But of course total salaries are also the sum of the amount of time worked time the unit salary, or

$$W = w\mathbf{l}'\mathbf{s} \tag{4.49}$$

From 4.47 and 4.40 it is obvious that

$$w = \omega \tag{4.50}$$

This means that the unit salary w is a number between zero and one that represent the share of Net Product which ends up in wages.

$$0 \leq w \leq 1 \tag{4.51}$$

Now, a first thing to notice is that the above system consists of n independent equations and $n+2$ variables. Even if we use the value of the Net Product as *numeraire*, we are left with $n+1$ variables, which leaves us with a degree of freedom. Since fixing the value of an other price would not have economic meaning, we are left with the choice of setting either w or r. After we have set one of the two , the system will be determined.

It is worth to spend some words on the strong economic meaning behind the fact that, in order to determine the system, we have to exogenously fix one of the distributive variable. It is an approach to distribution that is in sharp contrast with the neoclassical tradition, where the distribution arising from the operation of the market is a strictly technical matter, based on the diminishing marginal returns of each factor.
The fact that one distributive variable must be set outside of the production process opens space to a multitude of possibilities. For example it opens space to a competitive bargaining between workers and owners of the means of production over the surplus, taking a social and political tone. Finoia (1979) considers the case in which the part of net product going to workers is set via a collective labour agreement . An other possibility is that one of the variable may be fixed from other sectors of the economic system, for example the rate of profit may be strongly influenced by the monetary interest rate (Sraffa, 1960). In general, Sraffa's system is coherent with any distribution theory in which one variable is apt to be fixed from outside the production process (Pasinetti, 1988). This means that, even under conditions of thoroughgoing competition, the distribution of the social surplus remains *open and contestable* (Aspromourgos, 2013). Again, this is in sharp contrast with the Walrasian tradition in which wages and rate of profit are technically and uniquely determined in the production process, and it is by many considered one of Sraffa's sharpest critique to the neoclassical framework.

Notice that the fact that one of the distributive variables must be set from outside the production process must not be seen as a contradiction with our statement of before that the heterogeneity of capital imposes a simultaneous determination of prices and distribution of net product. In fact what could not be determined because of the heterogeneity of the commodities was the value of the circulating capital employed, to which the rate of profit, a pure number, has to be applied. It was therefore the *aggregate* distribution of the *value* of the net product (such that all industries received an uniform π) to be determined simultaneously with the prices. Taking as given the rate of profit does not solve the problem of the distribution of the value of net income *before* the determination of prices, since, without prices, we would not be able to know the value of the part of net income going to workers or owners of means of production. As a consequence of this, prices still cannot being determined. Therefore even setting w or π as fixed, we are still considering the two problems, distribution of income (determination of the remaining distributive variable) and setting of prices, as simultaneous. Kurz and Salvadori (2001)

point out that such an "asymmetric" treatment reserved to distributive variables (one fixed exogenously, and one determined endogenously) is a trait that Sraffa shares with Von Neumann's dynamic general equilibrium analysis (see section 5.3.2)

Before determining the general solution to the price system, let us consider, following Sraffa, firstly the two extreme cases, in which all the net product ends in the hands of one of the two social classes.

Maximum unit wage

Let us consider the case in which the whole net product goes to the workers. If we assume $\pi = 0$ and consider the case in which the Net Product is used as *numeraire* (so that we have $w = 1$ in this case) the system 4.44 become

$$\mathbf{p}'(\mathbf{I} - \mathbf{A}) = \mathbf{l} \tag{4.52}$$

Now let us consider at which conditions system 4.52 has an economically meaningful solution.

Proposition 4.3.1. *If the matrix* $\mathbf{I} - \mathbf{A}$ *has full rank and the maximum eigenvalue of* $\boldsymbol{A}$ *is smaller than one, the system 4.52 has an uniquely determined, non-negative solution, which is equal to*

$$\mathbf{p}' = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-\mathbf{1}} \tag{4.53}$$

Proof. The existence and uniqueness of the solution is straightforward from the hypothesis that $\mathbf{I} - \mathbf{A}$ has full rank.
The proof for the non-negativity is rather similar that the proof for Proposition 3.4.1: Because of proposition 10.2.4 of Mathematical Appendix, given the assumption on the eigenvalue, we have that of $(I - A)^{-1}$ is non negative . This, united with the positivity of $\mathbf{l}$, guarantees the non-negativity of the solution. □

Notice that if instead of setting the Net Product as *numeraire* we would have chosen a random commodity, solution 4.53 would simply become the analogous

$$\mathbf{p}' = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-\mathbf{1}} w \tag{4.54}$$

It is now worth to spend some words on the interpretation of the expression above. Recall from section 3.4 the interpretation of the coefficients of the matrix $(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}$: an element of $(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}$, α_{ij}, represent the amount of i that must be produced overall to obtain one unit of net product of j. Given our convention regarding the unit of measure, according to which total production of each industry equal to one, this amounts correspond to proportions of total quantity produced. In other words, coefficient α_{ij} represent the part of the total production of commodity i that concurs to the production of one unit of net product j.
If we left-multiply one column of $(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}$, α_j, by the vector of labor requirements $\mathbf{l}$, we sum the product of the part of the production of each industry i that concur to

produce one unit of net product of j, α_{ij}, and the part of total labor that is employed in the production of the total quantity of commodity i. Each element of the sum, $\alpha_{ij}l_i$, is therefore the portion of total labor employed in the production of the amount of commodity i necessary to produce one unit of net product of commodity j, and finally the scalar v_j

$$v_j = \mathbf{l}'\alpha_{\mathbf{j}} \tag{4.55}$$

represents the portion of total labor employed in the production of all commodities necessary for the production of one unit of net product of j, or *total requirement of labor*[13] for a unit of j.
Repeating the operation with all the columns of $(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}$ we obtain the vector of total requirements of labor[14]

$$\mathbf{v} = \mathbf{l}'(\mathbf{I} - \mathbf{A})^{-\mathbf{1}} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{4.56}$$

Now, compare the 4.56 with 4.53. It is clear that, if $\pi = 0$, then $\mathbf{p} = \mathbf{v}$, meaning that the price of a commodity i is *equal to the amount of labor employed for producing it.*[15]
"At no other level of wage", Sraffa (1960) writes, "do value follow a simple rule".
Notice that this simple rule is nothing more that the Ricardian-Marxian labor theory of value, according to which the source of value was the labor-time employed in its production. In Sraffa's framework, therefore, that theory represents only a special, and quite unrealistic, case.[16]

Maximum rate of profit

Let us now consider the opposite case, in which the profits are at their maximum level, which we denote

$$\pi = \Pi \tag{4.57}$$

[13] As opposed to l_j that is the *direct requirement of labor.*

[14] Pasinetti (1986) calls it vector of vertically integrated coefficients of labor

[15] Notice that if we take a random commodity as numeraire instead of the value of the Net Product, so that with $\pi = 0$ we do not have $w = 1$, expression 4.53 become $\mathbf{p} = \mathbf{l}'(\mathbf{I} - \mathbf{A})^{-\mathbf{1}}\mathbf{w}$, and so prices would be just *proportional* to necessary labor time.

[16] It is actually unfair to liquidate so easily the relationship between Sraffa and labor theory of value. Many authors, for instance, see in the double normalization 4.47 and 4.40 a recovery of such a theory, since it consists in setting as equal the labor employed and the value of the net product (Perri, 2014) (Gattei, 2011). Pasinetti (1986), on the other hand, sees in the subsystems mentioned in the Appendix the vision of an economy that basically uses labor as the only net input in order to produce the net product. Such a fascinating debate, nevertheless, is beyond the subject of this dissertation.

in correspondence of $w = 0$.
System 4.42 becomes then

$$\begin{cases} (a_{11}p_1 + a_{21}p_2 + \cdots + a_{n1}p_n)(1+\Pi) = p_1 \\ (a_{12}p_1 + a_{22}p_2 + \cdots + a_{n2}p_n)(1+\Pi) = p_2 \\ \vdots \\ (a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{nn}p_n)(1+\Pi) = p_n \end{cases} \tag{4.58}$$

or

$$\mathbf{p}'[\mathbf{I} - (1+\Pi)\mathbf{A}] = \mathbf{0} \tag{4.59}$$

The maximum rate of profit Π is equal to the value (at prices $\mathbf{p}$) of the Net Product divided by the value of the means of production employed.

$$\Pi = \frac{\mathbf{p}'(\mathbf{I} - \mathbf{A})\mathbf{s}}{\mathbf{p}'\mathbf{A}\mathbf{s}} \tag{4.60}$$

We may wonder about the economic meaning of an economy with wages equal to zero. This hypothesis would make more sense if we would consider, as mentioned above, the wage only as the part of the surplus that goes to workers, excluding the subsistence part, which would appear in the means of production. With this point of view, which treats "men as horses" and is more appropriate for the description of a slave-economy that a capitalistic one, the assumption of $w = 0$ would make more sense and the analysis conducted below would be identical.

Before going into the analytical solution, let us consider the simple example 4.30. The system for the determination of prices and Π, setting $p_i = 1$, would be

$$\begin{cases} (280p_g + 12p_i)(1+\Pi) = 575p_g \\ (120p_g + 8p_i)(1+\Pi) = 20p_i \\ p_i = 1 \end{cases} \tag{4.61}$$

which solved gives the results of $p_i = 15, \Pi = 0.25$. Consider what happen in the wheat industry. Of the 575 quarters of wheat produced, 280 quarters go to restore the original necessity of wheat, 180 quarters are sold in the market in order to obtain 12 tons of iron ($15 * 12$, meaning $p_i * q_i$), and 115 constitutes the profit. Since the means of production of the wheat industry has a value of 280+180=460 (measured in quarters of wheat), the profit is equal exactly to $\Pi = \dfrac{115}{460} = 0.25$.

Now, let us consider the conditions for which the system 4.59 is determined. Setting, for simplicity,

$$\lambda = \frac{1}{1+\Pi} \tag{4.62}$$

system 4.59 become

$$\mathbf{p}'[\lambda\mathbf{I} - \mathbf{A}] = \mathbf{0} \tag{4.63}$$

This is an homogeneous system, so that a necessary condition for the system to have non-zero solution is that the determinant of the matrix of coefficients, $[\lambda\mathbf{I} - \mathbf{A}]$, is equal to zero. Therefore we can find a vector of non-zero prices that solves the 4.59 only for values of λ (and connected values of Π) for which

$$Det[\lambda\mathbf{I} - \mathbf{A}] = 0 \tag{4.64}$$

Now, it appears from 4.64 that the values of λ that allow for a non-zero price vector are the *eigenvalues* of matrix $\mathbf{A}$. This could create a problem, since a (n, n) matrix has n eigenvalues (even if may be repeated), and so we appear to have n possible values of Π which guarantee a different solution to the system. But notice that, given that λ is an eigenvalue, from expression 4.63 is clear that p is the left-eigenvalue of $\mathbf{A}$ connected with λ. Therefore, thank to Proposition 10.2.3 (mathematical Appendix) we know that being $\mathbf{A}$ a non-negative matrix, there is only one eigenvalue that assures that the left-eigenvector (our price vector) has all non-negative elements, and it is λ_{max}, the maximum eigenvalue.[17]Since such eigenvector is our price vector, and it would not have sense to consider negative prices, we can impose the (economic) condition that

$$\lambda_{max} = \frac{1}{1+\Pi} \quad \Rightarrow \quad \Pi = \frac{1}{\lambda_{max}} - 1 \tag{4.65}$$

We have now to impose an other economic condition for the solution to have sense. In fact, beside the non-negativity of the price vector, we need to have also the non-negativity of the maximum rate of profit, Π. That is guaranteed by the following proposition:

Proposition 4.3.2. *If the economic system is viable (Property IIIbis), then*

$$\Pi = \frac{1}{\lambda_{max}} - 1 > 0 \tag{4.66}$$

Proof. In the first place, we know from Proposition 4.2.1 that

$$\lambda_{max} > 0 \tag{4.67}$$

Then, let $\mathbf{p}$ be the left eigenvector connected to λ_{max}. By definition, and right-multiplying by the sum vector

$$\begin{aligned} \mathbf{p}'\mathbf{A} &= \lambda_{max}\mathbf{p}' \\ \mathbf{p}'\mathbf{A}\mathbf{s} &= \lambda_{max}\mathbf{p}'\mathbf{s} \end{aligned} \tag{4.68}$$

[17]If $\mathbf{A}$ is also irreducible, then λ_{max} is actually the *only* eigenvector with all non-negative elements, and it is strictly positive.

From the condition of viability, on the other hand, pre-multiplying by $\mathbf{p}$

$$\begin{aligned}(\mathbf{I}-\mathbf{A})\mathbf{s} &> \mathbf{0} \\ \mathbf{s} &> \mathbf{A}\mathbf{s} \\ \mathbf{p}'\mathbf{s} &> \mathbf{p}'\mathbf{A}\mathbf{s}\end{aligned} \tag{4.69}$$

Confronting 4.68 and 4.69 we have that

$$\mathbf{p}'\mathbf{s} > \lambda_{max}\mathbf{p}'\mathbf{s} \tag{4.70}$$

which, given the non negativity of $\mathbf{p}$ and $\mathbf{s}$,

$$\lambda_{max} < 1 \tag{4.71}$$

Therefore we have that λ_{max} in restricted to 0 and 1, and for this value Π as defined above is always positive. □

We have now both the mathematical and economic conditions to prove that a set of pricing solving 4.59:

Proposition 4.3.3. *If the economic system is viable and* $\Pi = \frac{1}{\lambda_{max}} - 1$ *then system 4.59 admits at least one non-negative solution*

$$\mathbf{p} \geq \mathbf{0} \tag{4.72}$$

Proof. Follow from the discussion above: the solution $\mathbf{p}$ will be the non-negative eigenvector connected with the maximum eigenvalue λ_{max}. □

We may wonder now if this solution is unique. Being the determinant of the coefficient zero, the matrix of coefficients has not full rank, and an equation can be eliminated. If we assume that the remaining $n-1$ equations are linearly independent,

$$Rank[\mathbf{I}-(1+\Pi)\mathbf{A}] = n-1 \tag{4.73}$$

then we have a system of $n-1$ equation with n variables. Setting the price of a commodity (or the value of the net product) as one permits to show the following Proposition.[18]

Proposition 4.3.4. *Consider a viable economic system such that* $Rank[\mathbf{I}-(1+\Pi)\mathbf{A}] = n-1$. *Then,if we set* $p_i = 1$ *for some* i, *the system 4.59 has an unique solution.*

Proof. The system 4.59 has n-1 variables in n-1 independent equations, therefore is uniquely determined. □

[18]The uniqueness of the solution may be proved also replacing the assumption in 4.73 with the one, economically founded, that the maximum eigenvalue of matrix $\mathbf{A}$ i equal to the one of the sub-matrix composed by basic commodities only (Pasinetti, 1975). Moreover, in this way, Proposition 4.3.3 holds with strict inequality.

Let us consider again what we have done in this subsection. We have proved that if an economy is viable, i.e. is capable of producing a surplus, and $n-1$ equations of production are linearly independent, then we have an unique positive value for the maximum rate of profit and an unique set of relative prices. But let us not lose the contact with the point of view taken by Sraffa: he is analysing the intrinsic properties of a generic *actual* economy. The mathematical conditions are imposed because any economy worth of studying (in Sraffa's point of view) will actually be able to produce a surplus, and will need to have a positive maximum rate of profit, meaning that it will have to respect both the physical and economic conditions that permit to it to reproduce itself. Therefore any economy which actually can be analysed will *have* to respect the discussed mathematical properties, if an uniform rate of profit is paid in all sectors of the economy.

The general case

Let us now consider the most general case, for values of the rate of profit between its extreme cases, or

$$0 < \pi < \Pi \tag{4.74}$$

Recalling the discussion of the sections above, we can easily prove that, setting the value of one distributive variable and of one commodity (or the Net Product) as *numeraire*, then from the system 4.44 we can obtain an unique set of positive prices.

Proposition 4.3.5. *Consider a viable economic system. Then, given the value of either w or π and considering the convention 4.47:*

i) System 4.44 has, for $0 < \pi < \Pi = \frac{1}{\lambda max} - 1$, an unique solution $\boldsymbol{p}$

ii) This solution is strictly positive, or

$$\mathbf{p} > \mathbf{0} \tag{4.75}$$

Proof. Let us now prove the two statements of Propostion 4.3.5

i) for $\Pi = \frac{1}{\lambda max} - 1$ and $\pi < \Pi$ we have that

$$(1+\pi) < \frac{1}{\lambda_{max}} \tag{4.76}$$

Therefore we can apply Proposition 10.3.5 for non-negative matrices and have that the coefficient matrix of system 4.44 has an inverse, and that such inverse is non-negative.

$$[\mathbf{I} - (1+\pi)\mathbf{A}]^{-\mathbf{1}} \geq \mathbf{0} \tag{4.77}$$

Therefore the solution is uniquely determined in

$$\mathbf{p}' = w\mathbf{l}'[\mathbf{I} - (1+\pi)\mathbf{A}]^{-\mathbf{1}} \tag{4.78}$$

ii) for $0 < \pi < \Pi$ we have also $w > 0$. Since $\mathbf{l} > 0$ and considering expression 4.77, it follows that the unique solution $\mathbf{p}$ must be strictly positive

□

Looking at solution 4.78 we can underline the basic features of the theory of value arising from Sraffa's book: the value of the production depends on two factors: the *technical* conditions of production, represented by matrix $\mathbf{A}$ and vector $\mathbf{l}$; and on the *social* conditions of production the distributive variables w and π (Steedman, 1981). It is clear that it is a completely different theory of value compared to the neoclassical one, where prices are indicators of scarcity.

The role of basic commodities

We can at this point analyse in greater depth the different role of basic and non basic commodities in the determination of prices. If the methods of production of a basic commodity would change, this would have repercussion on the prices of *all* other commodities, and on the distributive variables too. This because, by definition, a basic commodity enters in the production process of all commodities, so a change in its method of production (that would change its value) would consequently affect (directly or indirectly) the value of the circulating capital used in all other industries. The same cannot be said for non-basic commodities. From the definition given in section 4.2, non basic-commodities are those commodities that enters only in their own method of production, in the methods of production of other non-basic commodities, or not even in their own production processes. They have therefore a mere passive role in the economy. If a non-basic commodity would disappear from the system (or, less drastically, would change production process) this would not have any effect on the prices of other commodities (apart, eventually, on the prices of other non basic commodities).
Consider the example 4.35. The system determining the prices would be

$$\begin{cases} (250p_g + 10p_i)(1+\pi) + l_g w = 400p_g \\ (120p_g + 8p_i)(1+\pi) + l_i w = 20p_i \\ (500p_g + 2p_i)(1+\pi) + l_w w = 100p_w \end{cases} \tag{4.79}$$

Now, we can see that the variable p_w appears only in the third equation. If we cancel the third equation, the system would still have two independent equations and two variables (having set the price of one commodity as *numeraire* and one of the distributive variables), and it would be satisfied by the same values than the system 4.79.

We can show this result analytically making use of the discussion over basic and non basic commodities made in the previous section. We have shown that a generic matrix of technical coefficients $\mathbf{A}$ with at least one basic commodity can be rearranged in order to obtain the form 4.36

$$\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{0} & \mathbf{A_{22}} \end{bmatrix}$$

Therefore the matrix of coefficients of system 4.44 can be rearranged in

$$
\begin{aligned}
[\mathbf{I}-(1+\pi)\mathbf{A}] &= \begin{bmatrix} \mathbf{I_r} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_{n-r}} \end{bmatrix} - (1+\pi) \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{0} & \mathbf{A_{22}} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{I_r} - (1+\pi)\mathbf{A_{11}} & -(1+\pi)\mathbf{A_{12}} \\ \mathbf{0} & \mathbf{I_{n-r}} - (1+\pi)\mathbf{A_{22}} \end{bmatrix}
\end{aligned} \tag{4.80}
$$

So that system 4.44 can be rewritten as

$$
\begin{bmatrix} \mathbf{p_1'} & \mathbf{p_2'} \end{bmatrix} \begin{bmatrix} \mathbf{I_r} - (1+\pi)\mathbf{A_{11}} & -(1+\pi)\mathbf{A_{12}} \\ \mathbf{0} & \mathbf{I_{n-r}} - (1+\pi)\mathbf{A_{22}} \end{bmatrix} = \begin{bmatrix} \mathbf{l_1'} & \mathbf{l_2'} \end{bmatrix} w \tag{4.81}
$$

or, multiplying the left side

$$
\mathbf{p_1'}[\mathbf{I_r} - (1+\pi)\mathbf{A_{11}}] = \mathbf{l_1'}w \tag{4.82}
$$

$$
-\mathbf{p_1'}(1+\pi)\mathbf{A_{12}} + \mathbf{p_2'}[\mathbf{I_{n-r}} - (1+\pi)\mathbf{A_{22}}] = \mathbf{l_2'}w \tag{4.83}
$$

where $\mathbf{p_1'}$ and $\mathbf{p_2'}$ are the vector of prices respectively of the basic commodities and non-basic commodities.
Now, if $\mathbf{A}$ is invertible, so are $\mathbf{A_{11}}, \mathbf{A_{12}}$ and $\mathbf{A_{22}}$. Then systems 4.82 and 4.83 solve in

$$
\mathbf{p_1'} = \mathbf{l_1'}[\mathbf{I_r} - (1+\pi)\mathbf{A_{11}}]^{-1}w \tag{4.84}
$$

$$
\mathbf{p_2'} = \mathbf{l_2'}[\mathbf{I_{n-r}} - (1+\pi)\mathbf{A_{22}}]^{-1}w + \mathbf{l_1'}[\mathbf{I_r} - (1+\pi)\mathbf{A_{11}}]^{-1}(1+\pi)\mathbf{A_{12}}[\mathbf{I_{n-r}} - (1+\pi)\mathbf{A_{22}}]^{-1}w \tag{4.85}
$$

From the above expressions appears clearly that the prices of the basic commodities, $\mathbf{p_1'}$, depends only on the methods of production of the basic commodities themselves, $\mathbf{A_{11}}$ and $\mathbf{l_1'}$. At the same time, the prices of non-basic commodities depends on the whole matrix of technical coefficients: $\mathbf{A_{11}}, \mathbf{A_{12}}, \mathbf{A_{22}}, \mathbf{l_1'}$ and $\mathbf{l_2'}$.

4.3.2 Relationship between prices and the distributive variables

Sraffa then investigates the relationship that the different variables of his system have with each other. He starts considering the effect that a change in one of the distributive variables would have on the price of the different commodities, produced with heterogeneous methods of production. To underlying the forces in action, he makes the following thought experiment: he considers as a starting point an economy in which all the Net Product is absorbed in wages. From here, he consider what would happen in the various industries if wages would decrease a bit, and positive profits would arise. Of course following a change in the distributive variables prices would change, but let us assume for one moment that they do not. Then it is clear that the decrease in salary would affect the balance sheet of the different industries differently: for example it is clear that an industry employing 1000 workers would save more from the decrease in salary than an industry employing 200 workers; and the same time, since the rate of profit must be paid uniformly across industries, an industry which, at the current (kept fixed) prices,

is is using means of production for the value of 1000, would have to pay an amount of profits ten times higher than an industry whose value of the means of production is 100. Therefore, an industry with a sufficiently low proportion of labor to means of production would have a deficit on its payments for wages and profits. Conversely, an industry with an high proportion would have a surplus (Sraffa, 1960). It follows that a change in price would be necessary in all industries to redress the balance. In particular, an industry in "deficit" would have to increase its prices relative to its means of production, in order to add the quantity of gross product available to pay the profits, and we would have the opposite tendency in an industry with a "surplus".
Unfortunately these "tendencies" cannot give us a rule to determine which prices will increase or decrese. Consider in fact an industry with an low proportion of labor to capital: from the discussion above it is a good "candidate" for a deficit and therefore an increase in price. But whenever this unbalance would occur does not depends only from its method of production, but also from the method of production of *all the industries producing its means of production.* Not only: it would depend also on the methods of production of the industries producing the means of production of its means of production, and so on. It may happen, for example, that the value of its means of production is *lower* after all the changes in prices, so lower to actually bring the industry in the group of the "high proportion" industries.

Let us now formalize this considerations. Consider expression 4.78. Following Pasinetti (1975) let us consider the effect on prices of a change in the rate of profit instead that a change in wage. Let us set $w = 1$.[19]In this way, the solution prices of 4.78 depend only on coefficients of $\mathbf{A}$ and on π. Let us write them as a function of π , so $p_i(\pi)$, and substitute them in the system 4.42. Price p_k would then appear as

$$p_k(\pi) = l_k + (1+\pi)\sum_{i=1}^{n} a_{ik}p_i(\pi) \tag{4.86}$$

Consider now the ratio between a generic price p_k and the price of the first commodity, p_1 , which consist in expressing the price of commodity k in terms of the first commodity

$$\frac{p_k}{p_1}(\pi) = \frac{l_k + (1+\pi)\sum_{i=1}^{n} a_{ik}p_i(\pi)}{l_1 + (1+\pi)\sum_{i=1}^{n} a_{i1}p_i(\pi)} \tag{4.87}$$

Taking the derivative of expression 4.87 with respect to π we can see that, at least in the neighbourhood of π, the sign of the change in the relative price of p_k with respect to p_1

[19]Notice that if we set $w = 1$ we have to specify that normalization 4.47 do not hold. If that were the case, setting $w = 1$ would mean to set $\pi = 0$, as showed in subsection 4.3.1. We are here simply using the unit wage, instead of a random commodity or the net product, as numeraire.

will depend on the sign of the following expression[20]

$$[l_1 \sum_{i=1}^{n} a_{ik}pi - l_k \sum_{i=1}^{n} a_{i1}pi] + (1+\pi)[l_1 \sum_{i=1}^{n} a_{ik}\frac{\delta p_i}{\delta \pi} - l_k \sum_{i=1}^{n} a_{i1}\frac{\delta p_i}{\delta \pi}] \tag{4.88}$$

The expression in 4.88 allow us to see what we can know about the effect of a change in π on the relative price of a commodity with respect an other commodity. Observe the parts in square brackets. The first expression assumes positive value whenever

$$\frac{l_1}{\sum_{i=1}^{n} a_{i1}pi} > \frac{l_k}{\sum_{i=1}^{n} a_{ik}pi} \tag{4.89}$$

This makes sense: whenever the ratio between labor and the value of the means of production is greater in industry 1, following the reasoning of the beginning of the section, commodity k would tend to develop a "deficit" with respect to commodity 1 (since its means of production would be worth relatively more), and therefore its price would have the tendency of increase relatively to p_1. We can call this effect *capital intensity effect*(Pasinetti, 1975), and it will be positive for all commodities with greater capital intensity of commodity 1 and negative for all commodities with lower capital intensity. The sign of the effect will therefore depend on the specific methods of production of commodity k.
On the other hand, the second part of the expression is much more complicated, and its sign depends on the effect of the change of *all* the commodities in the system. We can call this effect *price effect*, and we cannot determine its sign.
Therefore from expression 4.88 we can say that, in most cases, an increase in the rate of profit would result in an increase in the price of all commodities with a greater capital intensity than the commodity used as *numeraire*, but this is not sure at all: there may be commodities in which the price effect could be of the opposite sign with respect to the capital intensity effect, and so strong to reverse the sign of the change.

It is worth to point out also an other characteristic resulting from the discussion above. The fact that we are considering only a change in the price of a commodity *relative* to

[20]The numerator of $\frac{\delta(\frac{p_k}{p_1})}{\delta \pi}$ would actually be

$$l_1 \sum_{i=1}^{n} a_{ik}pi + (1+\pi)\sum_{i=1}^{n} a_{ik}pi \sum_{i=1}^{n} a_{i1}pi + l_1 \sum_{i=1}^{n} a_{ik}\frac{\delta p_i}{\delta \pi} + (1+\pi)^2 \sum_{i=1}^{n} a_{i1}pi \sum_{i=1}^{n} a_{ik}\frac{\delta p_i}{\delta \pi} -$$
$$-l_k \sum_{i=1}^{n} a_{i1}pi - l_k \sum_{i=1}^{n} a_{i1}\frac{\delta p_i}{\delta \pi} - (1+\pi)\sum_{i=1}^{n} a_{ik}pi \sum_{i=1}^{n} a_{i1}pi - (1+\pi)^2 \sum_{i=1}^{n} a_{ik}pi \sum_{i=1}^{n} a_{i1}\frac{\delta p_i}{\delta \pi}]$$

which,cancelling out the equal elements and considering that

$$\sum_{i=1}^{n} a_{i1}pi \sum_{i=1}^{n} a_{ik}\frac{\delta p_i}{\delta \pi} = \sum_{i=1}^{n} a_{ik}pi \sum_{i=1}^{n} a_{i1}\frac{\delta p_i}{\delta \pi}$$

would result in expression 4.88

the price of an other commodity, means that we cannot say much about how the conditions of production of the first commodity influenced the change in its price. Indeed expression 4.89 show us that what matters is the relative capital intensity with respect to the *numeraire* commodity, so that a change in one direction could well depend on the change in value of the *numeraire* commodity more that in the observed one. For sure, changing *numeraire* commodity, it would change the intensity and maybe also the direction of the change. It was the research of an "Invariable Measure of Value", capable of keeping its value following a change in distribution, that brought Sraffa to the construction of his Standard Commodity, a composite commodity that possesses exactly such a property.[21] The change in the price of any commodity, if measured in terms of the Standard Commodity, would represent only the change in value deriving from his own methods of production.[22]

We have, anyway, a result that we can prove with reasonable assumptions; that the price of all commodities, express in term of the unit wage w, would increase following an increase in the rate of profit. We express this in the following proposition

Proposition 4.3.6. *Consider a viable economic system. Then if we set* $w = 1$[23]

$$\frac{\delta p_i}{\delta \pi} \geq 0 \quad \forall \quad i = 1, ..., n \tag{4.90}$$

Proof. Setting $w = 1$ the solution 4.78 appear as

$$\mathbf{p}' = \mathbf{l}'[\mathbf{I} - (1+\pi)\mathbf{A}]^{-\mathbf{1}} \tag{4.91}$$

By Proposition 10.3.5, since in its economically meaningful range $(1+\pi) < \frac{1}{\lambda_{max}}$, we have that $[\mathbf{I} - (1+\pi)\mathbf{A}]^{-\mathbf{1}}$ is a continuous non-decreasing function of $(1+\pi)$. □

Notice that this of course does not tell us anything about movements in relative prices, since there is no way of determining which price will rise more than others. Proposition 4.3.6 give us also a hint in the relationship that must occur between the rate of profit and the unit wage, which is the subject of the following subsection.

[21] as Sraffa writes

> In such a world, where everything moves in every direction; [..] ; where the prices of commodities rise or fall, and we cannot express in simple words (or any words) the conditions under which they rise or fall; where ... one sympathises with Ricardo in his search for an 'invariable measure of value'. In a universe where everything moves we need a rock to which to cling to, a horizon to reassure us when we see a brick falling that it is not us who are going up, nor that we are falling when we see a balloon rising. (D3/12/52/16)

Quoted in Carter (2014)

[22] Since in this dissertation we are focusing on the circular nature of the production process, we will not deal with the construction of the Standard System and the Standard Commodity.

[23] Notice that we are *not* assuming that all the net product goes to workers here: the case of distribution completely in favour of workers coincides with the case of $w = 1$ *only if* we are using the Net Product as *numeraire*. Here, on the contrary, we are using the unit wage as a *numeraire* *instead* of the net product.

4.3.3 The inverse relationship between w and π

Let us now consider the relationship between the two distributive variables. It is worth considering first a special case, that is the one in which the capital intensity is equal among all sectors of production.[24] Calling α the proportion between labor and value of means of production common to all industries, this means imposing the condition

$$\alpha = \frac{l_1}{\sum_{i=1}^{n} a_{i1}p_i} = ... = \frac{l_n}{\sum_{i=1}^{n} a_{in}p_i} \tag{4.92}$$

Substituting $l_1 = \alpha(\sum_{i=1}^{n} a_{i1}p_i), ..., l_n = \alpha(\sum_{i=1}^{n} a_{in}p_i)$ in system 4.42 and pooling similar terms we obtain:

$$\begin{cases} (a_{11}p_1 + a_{21}p_2 + \cdots + a_{n1}p_n)(1 + \pi + \alpha w) = p_1 \\ (a_{12}p_1 + a_{22}p_2 + \cdots + a_{n2}p_n)(1 + \pi + \alpha w) = p_2 \\ \vdots \\ (a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{nn}p_n)(1 + \pi + \alpha w) = p_n \end{cases} \tag{4.93}$$

or

$$\mathbf{p}'[\mathbf{I} - (1 + \pi + \alpha w)\mathbf{A}] = \mathbf{0} \tag{4.94}$$

which, setting for simplicity

$$\lambda = \frac{1}{(1 + \pi + \alpha w)} \tag{4.95}$$

transform in

$$\mathbf{p}'[\lambda\mathbf{I} - \mathbf{A}] = \mathbf{0} \tag{4.96}$$

From the discussion in subsection 4.3.4 we know that it is possible to determine the system in an economically meaningful way only if λ is equal to the largest eigenvalue of $\mathbf{A}$, so we set

$$\lambda_{max} = \frac{1}{(1 + \pi + \alpha w)} \tag{4.97}$$

At the same time, for the same reasons expressed in subsection 4.3.4, we will have that also expression 4.65 must hold, so that

$$\lambda_{max} = \frac{1}{1 + \Pi}$$

Putting the two together and rearranging we have

$$\Pi = \pi + \alpha w \tag{4.98}$$

We have therefore a very simple, linear relation between the rate of profit and the unit wage.

[24] It is the case implicitly analysed by Ricardo and Marx.

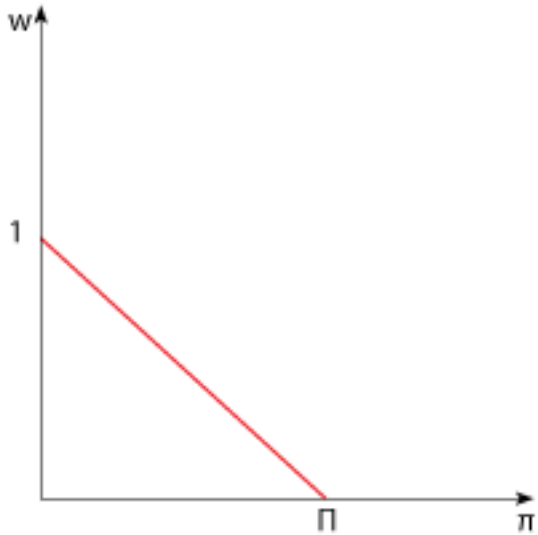
If we follow the the convention 4.47, as shown in subsection 4.3.2, the unit wage w assumes value between 0 and 1. In particular assumes value 1 whenever $\pi = 0$, so that from expression 4.98 we have[25]

$$\alpha = \Pi \tag{4.99}$$

Substituting expression 4.99 into the 4.98 and isolating π we obtain

$$\pi = \Pi(1 - w) \tag{4.100}$$

In conclusion, if the intensity of capital is equal among all sectors of production, then the relationship between unit wage and rate of profit is a linear one, which can be drawn as in the following figure



Unfortunately, if we remove the, very unrealistic, assumption of uniform intensity of capital, then the above relationship become much more complex.[26].

Consider the vector of solution prices 4.78. Let us set the price a random commodity i as *numeraire*, and let us select it from 4.78 with the use of the standard unit vector $\mathbf{e_i}$

$$p_i = 1 = w\mathbf{l}'[\mathbf{I} - (1 + \pi)\mathbf{A}]^{-\mathbf{1}}\mathbf{e_i} \tag{4.101}$$

The expression above indicates a polynomial implicit function of π and w, generally of order n. If we want to draw it on a graph it would have a much more complex form

[25]We can seen this also directly from system 4.93. If we set $w = 1$ and $\pi = 0$, or we set $w = 0$ and $\pi = \Pi$, the two systems differ only for α instead of Π.

[26]Sraffa actually showed that the linear relationship holds in the Standard System, and even in the actual system, provided that the wage measured in terms of the Standard Commodity is used as *numeraire*

than the linear on shown above. Moreover, its form would change accordingly to the commodity chosen as *numeraire*.
We can say just two things about the relationship described by 4.101: that, by construction, the intercept of the y-axis will be one and of the x-axis will be Π; that profit and unit wage have an inverse relationship, as shown by the following proposition

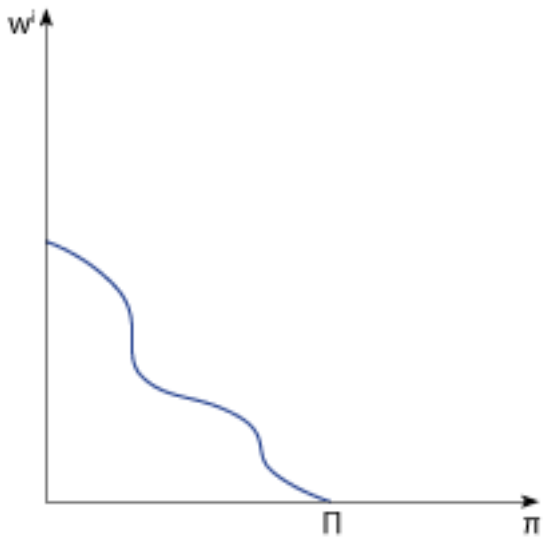
Proposition 4.3.7. *Consider a viable economic system. Then, whatever the commodity chosen as numeraire, the unit salary w is a monotonic non-increasing function of the rate of profit π.*

Proof. Consider the expression 4.101 and isolate the unit salary

$$w = \frac{1}{\mathbf{l}'[\mathbf{I}-(1+\pi)\mathbf{A}]^{-\mathbf{1}}\mathbf{e_i}} \tag{4.102}$$

Then, since $\mathbf{A}$ is non-negative and the economically significant values of π are such that $(1+\pi) < \frac{1}{\lambda_{max}}$, then $\mathbf{l}'[\mathbf{I}-(1+\pi)\mathbf{A}]^{-\mathbf{1}}\mathbf{e_i}$ is a non increasing function of $(1+\pi)$ because of 10.3.5 □

We present in the following image an example of the graphical representation of the distributive relationship between the unit wage w expressed in commodity i and the rate of profit π.



4.4 Conclusions

The work of Piero Sraffa, that we have introduced[27] in the previous sections, had at the time it went out quite an effect on the community of economists. Some scholar affirmed that, for the first time, the great classical problems of value and distribution were solved in a logically perfect way, within a circular process of production , implying heterogeneous production process among industries and comprehending a conflictual division of social surplus (Bortis, 2013).
Some others have been more critic: Hahn (1982) for instance tried to bring Sraffa's framework back to a specific case of the neoclassical model of General Equilibrium. In order to do so, anyway, he had to impose the condition of constant return to scale, that, as we have mentioned before, Sraffa refuses explicitly.

On this subject it is worth to spend some words, since in the last two sections we have considered the effect on a variable of a "change" in an other variable. As Sinha (2014) writes, Sraffa's proposition are not built on the usual mechanical cause and effect relationship. They do not contemplate any *actual* changes in the variables analysed. " All the dependence and changes of variables in Sraffa's propositions", Sinha writes, " describe logically necessary relationships between those variables, such as a change of 10 degrees of an angle in a Euclidian triangle must be associated with a 10 degrees combined change of the other two angles in the opposite direction."
As we have pointed out during the discussion above, Sraffa's prices represent the *necessary prices* through which the reproduction of the economic system can take place and an uniform rate of profit can be paid. They are not market prices (the demand is not considered in the analysis), as well they are not "costs", at least in the neoclassical terms. Sraffa considered the idea of calling his prices "cost of production", a term deriving from the classical tradition, but decided otherwise in order to not create confusion: in fact, beside the case of some non-basic commodities, the prices of the means of production of a commodity influence the determination of the price of that commodity not less than the price of that commodity influences the prices of its means of production. This is a consequence of the circular nature of the production system, in which commodities are produced by means of other commodities.

In order to understand better the meaning of Sraffa's prices, let us see what he writes himself in his unpublished papers [28]

"The significance of the equations is simply this: that if a man fell from the moon on the earth, and noted the amount of things consumed in each factory and the amount

[27] We have considered only half of Part One of his short but extremely dense book. The rest of Part One is dedicated to the construction of the Standard System and on the analysis of the features of the real economic system that with it could be enlightened, and on the reduction of values to dated quantities of labor. Part Two is dedicated on the analysis of Joint Production, and consider the use of Fixed Capital and Land as well. Finally, Part Three deals with the subject of the Switching of Techniques, that would have inspired the famous Cambridge Controversy over Capital.

[28] quoted in Gilibert (2006) from "Catalogue of Piero Sraffa Papers, edited by Jonathan Smith, Wren Library, Trinity College, Cambridge.

produced by each factory during a year, he would deduce at which values the commodities must be sold, if the rate of interest must be uniform and the process of production repeated. In short, the equations show that the conditions of exchange are entirely determined by the conditions of production."

Sraffa's prices, therefore, have to be considered "after the harvest" and "before the market" (after production and before final exchange in the commodity market) (Bellofiore, 2014), as specific properties of a generic, observed, economic system. Clearly the assumption of constant return to scale, in this view, is useless, since changes in quantities are not considered at all.

Chapter 5

Dynamic models

5.1 Introduction

In the previous chapters we have given little attention to the issue of the passing of time, and the few times that we have mentioned the subject, we have assumed that the economy was in a self-replacing, stationary state. In our analysis of Sraffa's model, in particular, the problem never arises, because of the intrinsically static nature of his work. Leontief, on the other hand, tried in his later work a generalization of his input-output model that allows for growth.

In the present chapter we will briefly introduce two analysis that deal with the question of a dynamic economy.
The first one consist, as we have already mentioned, in a generalization of Leontief's Open Model that we have presented in Chapter 3.
Subsequently, we will discuss a paper by John Von Neumann regarding the dynamic equilibrium of an economy, approached in a way that shares many features with the ones that we have dealt with in previous chapters. John Von Neumann has been a brilliant scholar, contributing substantially to the development of many fields in science, from pure mathematics to quantum mechanics. His fundamental contribution in economics is surely associated with the development of Game Theory as mathematical discipline. The paper we will consider has been originally published in Germany in Karl Menger's *Ergebnisse eines mathematischen Kolloqui*, and later on translated in english and published in the *Reviwe of Economic Studies* in 1945, on the initiative of Nicholas Kaldor. We will see how the model presented by Von Neumann has numerous resemblances with the approaches used by Leontief and Sraffa, even if it appears that the three scholars developed their works independently. In particular, Von Neumann's economy is characterized by the circularity of the production process, in a situation in which natural resources are not scarce and all commodities are employed in the production of all commodities. Following Champernowne (1945), we will focus on the economic interpretation of Von Neumann's model, omitting the rigorous mathematical proof, that, as the author himself points out, is interesting in itself.

One final consideration before dealing with the models. A satisfying treatment of dynamic economies should deal with a number of very complicated issues, in particular it would need a theory of growth capable of *explaining* the evolution of the consumption and investment decisions and of technological advancement. The models that we consider in this Chapter do not deal with such issues, and therefore it would maybe be better to consider them as "quasi-stationary" models.

5.2 Dynamic Leontief Model

In his later works Leontief attempted to introduce time into his analysis, in order to create a dynamic model with predictive capacity (Leontief *et al.*, 1953). In order to do so, he implements his static analysis with the introduction of a matrix of capital stock coefficients $\mathbf{K}$

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{bmatrix} \tag{5.1}$$

where the single element k_{ij} represent the technologically determined stock of fixed capital commodity i that industry j employs for unit of its output (Leontief, 1986). Leontief's set of linear difference equations representing input-output relationship in a growing economy would be

$$\mathbf{q_t} = \mathbf{A}\mathbf{q_t} + \mathbf{K}(\mathbf{q_{t+1}} - \mathbf{q_t}) + \mathbf{y_t} \tag{5.2}$$

where the notation is the same than in Chapter 3. The balancing assumption is that a good added to the stock in year t is put to use in year $t+1$.
Leontief then, taking as given the final demand of the last year y_t and the initial level of production q_0 is able to describe the total level of the output of each year making use of the recursive computation, rewriting equation 5.2 as

$$\mathbf{q_t} = \mathbf{K^{-1}}[(\mathbf{I} - \mathbf{A} + \mathbf{K})\mathbf{q_0} - \mathbf{y_1}] \tag{5.3}$$

This "Dynamic Leontief Inverse", as has been called, is not devoid of problems: indeed, the possibility of instable growing path (leading to exponential growth or, in the opposite case, even negative output) has not been ruled out by reasonable assumptions.

Anyway, we will not follow Leontief in this formalization, and we will stick to the assumption used in the previous chapters of economies employing only circulating capital. Notice, however, that a much more efficient way of treating fixed capital than the one used by Leontief can be found in Sraffa (1960) and in Neumann (1945), where fixed capital is transformed in circulating capital with the application of the formalization for joint production.

Following Pasinetti (1975), instead, we will briefly present a generalization of Leontief Open Model which employs only circulating capital, as in the rest of this dissertation.[1] Pasinetti consider the case in which the final demand increase at a constant rate because of the constant increase of the population. Consider system 3.19 for year t

$$(\mathbf{I} - \mathbf{A})\mathbf{q_t} = \mathbf{y_t} \tag{5.4}$$

$$\mathbf{l}'\mathbf{q_t} = \mathbf{L_t} \tag{5.5}$$

Since we consider an increase in production, it is better to differentiate between the components of final demand y_t. We distinguish between that part that goes int consumption, C_t and that part that goes int investment, J_t. We consider the latter as given, while the second one to be endogenously determined. Expression 5.4 become therefore

$$(\mathbf{I} - \mathbf{A})\mathbf{q_t} - \mathbf{J_t} = \mathbf{C_t} \tag{5.6}$$

In particular we assume, as mentioned above, that the population grow at a given, fixed percentage rate that we will call g. If N_t is the population at time t we have, therefore

$$N_t = (1+g)N_{t-1} = (1+g)^t N_0 \tag{5.7}$$

We assume that preferences of consumers stay stable over time, so that if the vector $C_.$ changes it is only due to the increase of the population. We can actually define the (time-invariant) per-capita consumption vector $\mathbf{c}$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \frac{1}{N_t}\mathbf{C_t} \tag{5.8}$$

so that we have

$$\mathbf{C_t} = \mathbf{c}N_t = (1+g)^t N_0 \tag{5.9}$$

Now if we make the assumption of constant return to scale, if the demand grow at a rate g per year, also the amount of inputs necessary to satisfy such an increased demand must grow at the same rate (Pasinetti, 1975). We have therefore determined our equilibrium equation for the investments in the following

$$\mathbf{J_t} = g\mathbf{A}\mathbf{q_t} \tag{5.10}$$

We can therefore rewrite the 5.4 as

$$(\mathbf{I} - \mathbf{A})\mathbf{q_t} - g\mathbf{A}\mathbf{q_t} = \mathbf{c}N_t \tag{5.11}$$

[1] Also Kurz and Salvadori (2000) consider a simplified version of Leontief dynamic model with only circulating capital.

which has an immediate solution

$$\mathbf{q_t} = [\mathbf{I}-(1+g)\mathbf{A}]^{-1}\mathbf{c}N_t \tag{5.12}$$

f the matrix $[\mathbf{I}-(1+g)\mathbf{A}]^{-1}$ is invertible (see section 3.4).
Notice that, apart from $N_t = (1+g)^t N_0$, all elements of the solution are time-invariant. This means that also the physical quantities increase at the constant rate $(1+g)$, denoting a dynamic equilibrium. The *scale* of the operations is eventually determined by the size of the actual population, but the proportions between industries will stay constant over time. For this reason we can divide both sides of 5.12 by N_t and obtain the two following expressions, representing the equations of the constant proportions denoting the equilibrium and their solution

$$[\mathbf{I}-(1+g)\mathbf{A}]\mathbf{q} = \mathbf{c} \tag{5.13}$$

$$\mathbf{q} = [\mathbf{I}-(1+g)\mathbf{A}]^{-\mathbf{1}}\mathbf{c} \tag{5.14}$$

Notice that the matrix $[\mathbf{I}-(1+g)\mathbf{A}]^{-\mathbf{1}}$ is a form of Leontief Dynamic Inverse.

5.3 Von Neumann's Model

In his very influential paper of 1937, Von Neumann presents his results concerning the long-term positions of a competitive system characterized by an uniform rate of return, in an inter-sectoral framework in which the production process is conceived as a circular flow (Kurz and Salvadori, 2001). The aim of his investigation is to determine, in an economy characterized by joint production and a great number of available techniques, the *relative* quantities and *uniform* rate of growth, and simultaneously the *relative* prices and *uniform* rate of profit which can be hold constant in time (and therefore represent an *equilibrium*). As pointed out above, such a model can barely be called dynamic, since it assumes invariance of a great number of economic variables which, instead, are likely to change over time, like consumption choices or technological knowledge (Pasinetti, 1975). Therefore is maybe more appropriate to talk about "quasi-dynamic" or "quasi-stationary" model (Champernowne, 1945).
Let us now present his model, and make order in the assumptions he made regarding the functioning of the production process, of the distribution process and on the characterization of his equilibrium.

5.3.1 The production process

Von Neumann consider an economy in which the production of a finite number n of commodities takes place in discrete time intervals. It is assumed that natural factors can be expanded in unlimited quantity, so that *scarcity* in neoclassical terms is not an issue. Unlike the model we have considered in the previous chapters, we will, initially, consider the case in which an industry is not limited to the production of a single commodity -

i.e: we will consider and economy with Joint Production. We consider that there exist m different methods of production for the production of the n commodities, with the possibility that $m > n$. A method of production, or *process*, is a function $\mathbb{R}^n \to \mathbb{R}^n$ which associates to a certain quantity of commodities (inputs) an other certain amount of commodities (outputs). Let q_{ij} be the amount of commodity i used in the $j-th$ production process, and Q_{ij} the amount of commodity i produced by the $j-th$ process.[2]Then define $\mathbf{q_j}$ be the vector containing all the quantities used in process j, and $\mathbf{Q_j}$ all the quantities produced in the same process. Process F_i can therefore be represented as

$$F(\mathbf{q_i}): \quad \mathbf{q_i} \to \mathbf{Q_i} \tag{5.15}$$

Notice that we are, again, considering an economy employing only circulating capital and implicitly assuming that all production process have the lame length . Any way, as mentioned by Neumann (1945) and more rigorously treated by Sraffa (1960), we can always bring back the case of an economy using fixed capital to the caso of an economy with joint production employing only circulating capital, and any production process longer than the period of time we have chosen to consider can be broken down into shorter processes of appropriate duration (eventually adding intermediate goods as new commodities).
An important assumption made by Von Neumann is the one regarding Constant Return to Scale. As we have already seen, this means that increasing by a common factor the inputs used in the production process, the resulting output would be multiplied as well for the same common factor, or that, for every process F_i

$$\forall \alpha \in \mathbb{R} \quad F(\alpha \mathbf{q_i}) = \alpha F(\mathbf{q_i}): \quad \alpha \mathbf{q_i} \to \alpha \mathbf{Q_i} \tag{5.16}$$

An important point is that not every process available must be used in the actual economy, and that different processed may be used in different scales at different times. We call $x_i(t) \geq 0$, for $i = 1, ..., m$ the "intensity" (or scale) associated with the $i-th$ production process at time t. Given the constant return to scale assumption, we can divide all the elements q_{ij} and Q_{ij} for the scale of the process, and obtain the coefficients a_{ij} and b_{ij}, which represent the amount of commodity i respectively used and produced by the $j-t$ production process operating at scale $x_j = 1$. In this way the m-dimensional vector $\mathbf{x_t} = (x_1(t), ..., x_m(t))$ has a similar role of that one which was denoted $\mathbf{q}$ in previous chapters, while the time-invariants vectors $\mathbf{a_i}$ and $\mathbf{b_i}$ can be gathered into the (n, m) matrices $\mathbf{A}$[3] and $\mathbf{B}$. The complete production process at time t can therefore be represented in the following way

$$\mathbf{A}\mathbf{x_t} \to \mathbf{B}\mathbf{x_t} \tag{5.17}$$

Von Neumann imposes also a *feasibility constraint.* Indeed, the amount of inputs used in period t cannot be greater, of course, than the outputs produced in period $t-1$. This

[2]Notice that, of course, in the case of single-production we just had Q_j, since only one kind of commodities were produced.

[3]Notice that this is not the usual matrix of technical coefficients we have used in previous chapters, which was nevertheless a specific case of this one for the single-product production, with $m = n$

means imposing the following condition to the economy

$$\mathbf{Bx_t} \geq \mathbf{Ax_{t+1}} \tag{5.18}$$

or, focusing on the single commodity i

$$\sum_{j=1}^{m} b_{ij}x_j(t) \geq \sum_{j=1}^{m} a_{ij}x_j(t+1) \quad i = 1, ..., n \tag{5.19}$$

and it is the obvious generalization for an economy with joint production of the assumption of an economy in self-replacing state that we have considered in the case of single-product commodity.

Von Neumann then makes an important assumption regarding the determination of price, which we can call the Rule of Free Good. It states basically that if a product is overproduced, its price will be set equal to zero. This means that if for a commodity i expression 5.19 holds with strict inequality $>$ (so that the amount of i produced in t is strictly greater than the amount of i used in $t+1$), then we would have $p_i(t+1) = 0$.[4] This condition can be imposed by the following slackness condition, where $\mathbf{p_{t+1}}$ is the n-dimensional vector of prices at time t

$$\mathbf{p'_{t+1}}(\mathbf{Bx_t} - \mathbf{Ax_{t+1}}) = \mathbf{0} \tag{5.20}$$

meaning that either $p_i(t+1) = 0$ or $\sum_{j=1}^{m} b_{ij}x_j(t) - \sum_{j=1}^{m} a_{ij}x_j(t+1) = 0$, for $i = 1, .., n$.

5.3.2 The distribution Process

Von Neumann then makes some important assumptions regarding the distribution process.
First of all, it assumes wage to be fixed and at the subsistence level, in continuity with the classical tradition (Kurz and Salvadori, 2001), as in the special case Sraffa examines in subsection 4.3.4. All the net product ends up therefore in the hands of the owners of the means of production, and Von Neumann makes the further assumption that all such net product is reinvested in increasing tomorrow's circulating capital. As we have mentioned before, Kurz and Salvadori (2001) point out that this *asymmetrical* treatment reserved to the distributive variables, one exogenous and one endogenous to be determined simultaneously with prices, is similar to the one that Sraffa adopts in his book (even if Sraffa, at a certain point, prefers to set as exogenous variable the rate of profit). An other important assumption is that the rate of return of each active industry must be equal - i.e.: there cannot be super-profits in the economy. Indicating with Π_t the uniform rate of return at time t,[5] the condition can be written as

$$\mathbf{p'_t B} \leq (1 + \Pi_t)\mathbf{p'_t A} \tag{5.21}$$

[4] Notice that of course price are assumed to be non-negative, so $p_i \geq 0$.

[5] We use the same notation of the maximum rate of profit if Sraffa, since Von Neumann assumes (surplus) wages equal to 0.

or, focusing on a single industry i

$$\sum_{j=1}^{n} p_j(t)b_{ji} \leq (1+\Pi_t)\sum_{j=1}^{n} p_j(t)a_{ji} \tag{5.22}$$

which means that the revenues of industry i cannot be greater than the costs of industry i (including the uniform rate of profit).
Von Neumann then makes a final assumption related to the uniformity of the rate of return, which we can call the Rule of Idle Activities. If a production process is such that at current prices the firm using it would incur in a loss, then the process is not used. This means that for any industry i for which expression 5.22 holds with strict inequality $<$, we would have $x_i(t) = 0$. Also this condition can be imposed through a slackness condition of the form

$$[\mathbf{p_t'B}-(1+\Pi_t)\mathbf{p_t'A}]\mathbf{x_t} = \mathbf{0} \tag{5.23}$$

so that either $\sum_{j=1}^{n} p_j(t)b_{ji} - (1+\Pi_t)\sum_{j=1}^{n} p_j(t)a_{ji} = 0$ or $x_i(t) = 0$. The above conditions may be named together *Profitability Rule*. Von Neumann states that only those processes that, at current prices and rate of return, yield zero super-profits will be used, and those will be the most profitable ones (Champernowne, 1945).

5.3.3 Equilibrium characterization

Von Neumann is interested in "those states in which the whole economy expands without a change in structure" (Neumann, 1945), meaning that all industries in the economy grow at the same rate. In this way the proportions $x_1 : x_2 : ... : x_m$ would stay invariant even if the various x_i may change. We can cal g_t this common rate of expansion for time t. Anyway, as mentioned before, Von Neumann wants to characterize an equilibrium in which both the rate of return and the rate of growth are stable over time, so that $\Pi_t = \Pi_{t+1} = \Pi$ and $g_t = g_{t+1} = g$. We can then write

$$\mathbf{x_{t+1}} = (1+g)\mathbf{x_t} \tag{5.24}$$

In general we can drop the time indexes altogether, since the conditions characterizing the equilibrium must hold in any period.

Summarizing the discussion so far, the Von Neumann equilibrium must be characterized by the following equations, which are the updated version respectively of expressions 5.18, 5.20, 5.21 and 5.23:

i)

$$\mathbf{Bx} \geq (1+g)\mathbf{Ax} \tag{5.25}$$

ii)

$$\mathbf{p}'[\mathbf{B}-(1+g)\mathbf{A}]\mathbf{x} = \mathbf{0} \tag{5.26}$$

iii)

$$\mathbf{p}'\mathbf{B} \leq (1+\Pi)\mathbf{p}'\mathbf{A} \tag{5.27}$$

iv)

$$\mathbf{p}'[\mathbf{B}-(1+\Pi)'\mathbf{A}]\mathbf{x} = \mathbf{0} \tag{5.28}$$

v) [6]

$$\sum_{i=1}^{m} x_i > 0 \quad , \sum_{i=1}^{n} p_i > 0 \tag{5.29}$$

Now, without further assumptions about matrices **A** and **B** there will, in general, no solution satisfying conditions (i)-(v). For this reason Von Neumann added the further (questionable) assumption that any process must use or produce at least a bit of every commodity, or

$$a_{ij} + b_{ij} > 0 \quad \forall i, j \tag{5.30}$$

Von Neumann justifies this assumption saying that even very small quantities of commodities would make assumption 5.30, hold, so that it is not too unrealistic. Nevertheless, Kemeny *et al.* (1956) tried to generalize Von Neumann's model by imposing the two much weaker conditions that:

a) Every process must use some input. This means that every column of **A** must contain at least a positive component, or

$$\sum_{i=1}^{n} a_{ij} > 0 \quad \forall j = 1, ..., m \tag{5.31}$$

b) Every commodity can be produced in the economy. This means that every row of **B** must have at least one positive entry, or

$$\sum_{j=1}^{m} a_{ij} > 0 \quad \forall i = 1, ..., n \tag{5.32}$$

This generalization nevertheless has some problems. First of all, unlike assumption 5.30, does not guarantees that the economy will produce at least something of value. Therefore a sixth condition was added to Von Neumann's five:

vi)

$$\mathbf{p}'\mathbf{B}\mathbf{x} > \mathbf{0} \tag{5.33}$$

Moreover, relaxing the condition leads to the possibility of multiple equilibria, something that was instead avoided with assumption 5.30.

Let us consider now Von Neumann's solution (Neumann, 1945)

[6] since a solution with $x_1 = .. = x_m = 0$ or $p_1 = .. = p_n = 0$ would be economically meaningless.

Proposition 5.3.1. *At least a solution* $\boldsymbol{x}^*$, $\boldsymbol{p}^*$, g^* *and* Π^* *respecting conditions (i)-(v) exists.*
Moreover, for each solution

$$(1 + g*) = (1 + \Pi^*) = \alpha = \frac{\mathbf{p}^*\mathbf{B}\mathbf{x}^*}{\mathbf{p}^*\mathbf{A}\mathbf{x}^*} \tag{5.34}$$

Proof. Omitted □

In other words, given all the assumptions above, a solution for uniform rate of return and constant coefficient of expansion always exists. Moreover, the uniform rate of return and the coefficient of expansion of the economy are equal and *uniquely* determined by the technical possible processes $F_1, ..., F_m$.
Notice that the equilibrium is therefore unique with respect to the rate of return and the rate of expansion, but not with respect to vectors $\mathbf{x}^*$ and $\mathbf{p}^*$. This because just the relative proportions between the elements of $\mathbf{x}$ and $\mathbf{p}$ are sufficient to be a solution to the system, so that one can set intensity and price levels in any meaningful way. Von Neumann chose to normailze the sum of all prices and intensities to one, so that (Zalai, 2004)

$$\sum_{i=1}^{m} x_i = 1, \quad \sum_{i=1}^{n} p_i = 1 \tag{5.35}$$

Moreover there is an other important result obtained by Von Neumann: the value α representing the unique value of both rate of return and rate of expansion is equal to the *highest possible growth rate* and, at the same time, to the *smallest possible uniform rate of return* (Zalai, 2004). Formally

Proposition 5.3.2. *Let* $\boldsymbol{x}^*$, $\boldsymbol{p}^*$, $g^* = \Pi^* = \alpha - 1$ *be the solutions of proposition 5.3.1. Then*

$$g^* = \max\{g : \quad \mathbf{B}\mathbf{x} \geq (1 + g)\mathbf{A}\mathbf{x}, \quad \mathbf{x} \geq 0\} \tag{5.36}$$

$$\Pi^* = \min\{\Pi : \quad \mathbf{p}'\mathbf{B} \leq (1 + \Pi)\mathbf{p}'\mathbf{A}, \quad \mathbf{p} \geq 0\} \tag{5.37}$$

Proof. Omitted. □

So, what is Von Neumann saying with that results? It is saying that, because of competition among industries, it will be adopted the greatest rate of expansion possible, equal, of course, to the smallest rate of expansion f any good in a determined system. This can be rationalized by the fact that, if any other system with a greater rate of expansion would be available, any firm could adopt it and make positive super-profits, contradicting equilibrium condition (iii) (Champernowne, 1945).

5.3.4 The maximum rate of growth

Notice that if we consider the specific case of single-product industries, it is easy to express Von Neumann problem in terms of the analysis we have conducted in previous

chapters. In fact, in the case of single-product industry, for each vector price and rate of profit, it is always possible to determine the most efficient between two processes of production. This means that, for each commodity, there will be *only one* process of production with non-zero intensity. Choosing any time the unity of measure in such a way that the output matrix coefficient be an identity matrix, the choice of processes will therefore result in the choice between square input coefficient matrices

$$\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, ... \tag{5.38}$$

each of which will have as columns a different combination of the m available production processes.

Consider now for a moment expression 5.13 of the previous section in the case assumed by Von Neumann, meaning that the consumption is equal to zero (consumption of the worker is included into the coefficient matrices), so that all the surplus is reinvested:

$$[\mathbf{I}-(1+G)\mathbf{A}]\mathbf{q} = \mathbf{0} \tag{5.39}$$

We have denoted G the greatest possible rate of growth of the economy, corresponding at $\mathbf{c} = \mathbf{0}$. Analogously as what we have seen in subsection 4.3.5 with respect to the highest rate of profit in a Sraffian system, we know that the above homogeneous linear system will have non-zero solution only if the determinant of the coefficient matrix is equal to zero.

$$Det[\mathbf{I}-(1+G)\mathbf{A}] = \mathbf{0} \tag{5.40}$$

Again as in subsection 4.3.5 we know that this means that $(1+G)$ must be one of the eigenvalues of $\mathbf{A}$. More specifically, it will have to be the highest eigenvalue λ_{max}.

$$G = \frac{1}{\lambda_{max}} - 1 \tag{5.41}$$

Incidentally, this means also that G is equal to Π, the highest rate of profit, since λ_{max} is unique.
That said, we can therefore reformulate Von Neumann problem as the choice of the coefficient matrix among

$$\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, ...$$

to which is associated the lowest maximum eigenvalue

$$\lambda^1_{max}, \lambda^2_{max}, \lambda^3_{max}, ... \tag{5.42}$$

Proposition 5.3.1 and 5.3.2 will assure that it will be the solution of the problem and equal to the highest uniform rate of profit.

5.4 Conclusions

As we have seen, there are many contact points between the different approaches of Leontief, Von Neumann and Sraffa. First of all, the the modelling of an economy in which commodities are produced by means of commodities, and scarcity of natural resources is not an issue. This means a vision of the economy as a purely circular process.
Then, we have what we may call an "objectivist" approach in the determination of "equilibrium" values. The three scholars either explicitly reject (in the cases of Leontief and Sraffa) of simply ignores (Von Neumann) the role of psychology and subjectivism in the determination of prices, which, indeed, are mainly influenced by the material conditions of production. In the three approaches, prices are more similar to indicators of system necessities that of marginal utility or scarcity.
Moreover, the three thinkers share what Neumann (1945) called the "remarkable" duality in the determination of prices and scales, which are basically considered as separated phenomenon (this is less explicit in Sraffa, since he does not deals with the determination of quantities).
Finally, they all have been influenced by the works of classical economists, in contrast (more or less explicit) with some features of the dominant economic approach of their time.

Lenotief and Von Neumann anyway , differently than Sraffa, approached the subject of the evolution of an economy, while Sraffa was interested only in "such properties of an economic system that do not depend on changes in the scale of production or in the proportions of factors" (Sraffa, 1960). In order to do so, anyway, they had to assume the very restrictive property of constant return to scale.

In the following chapter we will consider some agent-based models based on the idea of a circular process of production, where commodities are produced by means of other commodities (and labor), and we will consider the insight deriving from our theoretical review

Part II

Simulation

Chapter 6

Model One: Purely physical model

6.1 Introduction

In this first, simple model we try to enlighten the economic consequences of the circularity of the production process, abstracting from the more complex issues of price determination and distribution among social classes. The only determinants of the functioning of the economy will therefore be the interdependencies among the various industries, the scale at which the various industries will operate, and a random factor that will arise from the chosen method of replacement of used inputs.

As in Neumann (1945) we assume that natural factors of production can be expanded in unlimited quantity (these include, for the moment, labor). This is coherent with the tradition of classical economists, for whom "natural" scarcity was a special case with respect to the more general one of "producible" resources, commodities used in the production of other commodities. The scarcity arising from the model will therefore be only "endogenous", that is determined by the technological requirements and by the proportions between the various industries.

The purely "physical" nature of this model will allow us to underline the role of physical requirements of the system. At the same time, the decentralized nature of the actions will determine situations which may differ from "optimal" ones (which would be better found via analytical tools), and at the same time gives a certain degree of variability and heterogeneity to the results.

6.2 General description

We consider an economy composed by just three commodities, which we have chosen to be "iron", "coal" and "wheat". The agents of the model are production units, which we have denoted as `firms`, which are able to produce some units of just one kind of commodity (we are therefore in the case of single-good production, as analysed in the previous chapters). In order to do so, they have to consume a certain amount of the

commodity they produce and a certain amount of other commodities. A clear economic requirement is that a firm that produces an amount X of commodity i cannot employ in its production process an amount of commodity i greater than X, otherwise such a technique of production would clearly be unsustainable. On the other hand, it is not required that every commodity employs any other commodity in its production, and the amounts required may variate greatly.

We start looking at our economy at the end of a production process. Each firm has consumed all its inputs and has only one kind of commodities. We assume that at the beginning each firm operates at a scale of one, i.e produces only one unit of commodity. Each firm, then, faces therefore the problem of recovering the inputs necessary to restart the production process. Since we have decided to abstract from the determination of prices, the inputs cannot be bought on the market, so we have to determine an alternative way to distribute the production. We assume that the economy in question represents a kind of "primitive communism", in which each firm makes available to the community the part of its production that it does not use. Therefore each firm, in a random order, subtracts from the unit produced the amount it will need the next period and tries to find the other inputs it needs by simply asking the other firms for it.

It is possible, anyway, that a firm may not be able to find the necessary means of production by the other firms. Finding the necessary inputs of commodity j, say, will depend on many factors: the amount the firm requires, the number of firms producing commodity j, but also the random order according to which the firms look for replacements. In fact it holds the principle "first come first served", and if the total quantities produced of commodity j is not sufficient to meet the total requirements of commodity j some firms will not receive what they need. If a firm is not able to replace its means of production, it will become inactive and stop producing from the subsequent period. The absence of prices in this process allow us to underline the physical nature of the requirements of the system, which may or may not be satisfied given the coefficients and the relative proportions of industries.

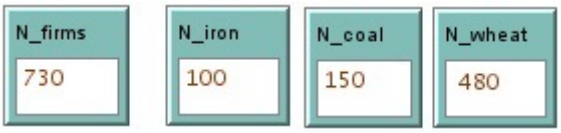
It is possible, on the other hand, that the economy produces more that what is strictly necessary to reproduce itself. Again, the possibility of generate a surplus will depend on the techniques of production of the various industries, and the actual generation of a surplus on the proportions among various industries. Such a surplus may be consumed or may be invested in an increased production for the following year. If it were to be consumed integrally, the economy would enter in a stationary state and the same quantities would be produced year after year. We have decided to analyse the opposite possibility that the surplus is completely reinvested, at least up to the feasible level, into new means of production. For "feasible level" we mean that, given the interdependencies of the method of production of the commodities, it will never be possible to consume *all* the commodities which form the surplus: the composite surplus will be used up to the point at which the available surplus of one commodity will be smaller than the smallest

amount required for the production of one unit of any commodity. To make an example, if the requirements of wheat for the production of one unit of iron, one of coal and one of wheat are respectively $1/3, 1/5$ and $1/2$, once the available surplus of wheat is smaller than $1/5$ it will not be possible to produce anything, regardless on the availability of surpluses of iron and coal. Also the investment phase is characterized by a random order according to which the different firms will check if there is enough surplus left for them to increase their production in the following year. Such increase in production is done one unit at the time, and, analogously to the theoretical models of Leontief and Von Neumann we assume, for now, constant return to scale.
Analogously to the "research of inputs" phase described above, also in this case the absence of prices allow us to investigate the physical constrains of the economy.

Let us now describe the various procedures of our code.

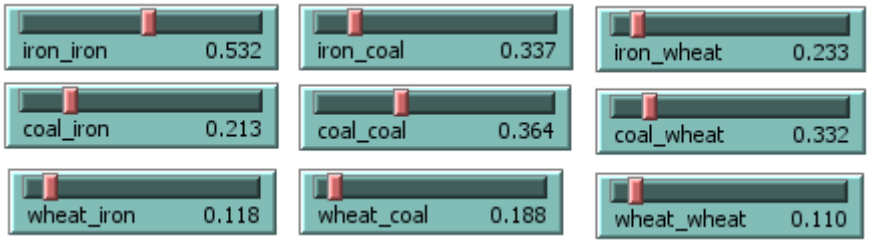
6.3 Setup

In this model we have just one kind of agents, `firms`. Each firm has the following built-i variables: `sector` is a string variable determining which kind of commodity the firm is able to produce; `iron_need`, `coal_need` and `wheat_need` indicate the amount of the various commodities that the firm necessitates to operate at the current scale of production; `iron_have`, `coal_have` and `wheat_have`, on the other hand, indicate the amount of the various commodities currently owned by the firm; `s` indicates the *scale* at which the firm is operating, which is also the amount of units it will produce; `output`, on the other hand, is the amount of commodity produced currently owned by the firms. There are other build-in variables, but they are instrumental to the functioning of the code and do not have particular economic relevance, so we omit to list them. In the following pages we will report only the part of the code referring to the firms which will receive `iron` as a sector, since for the firms in the other two sectors would be redundant. We have also some Global Variables: `surplus_iron, surplus_coal` and `surplus_wheat` indicate the amount of the three commodities which are left once all the firms have tried to replace their inputs; the variables `N_iron, N_coal` and `N_wheat` indicate the number of firms that will be created in each industry, while `N_firms` indicates the total number of firms, and can be set in the interface.



On the interface is also possible to set the coefficients of production of the economy, that is, as we have defined in chapter 2, the amount of commodity i necessary to produce a unit of commodity j. Following this notation, `iron_iron` will be the amount of iron necessary to produce an unit of iron, `coal_iron` will be the amount of coal necessary to produce one unit of iron, and so on. As we have mentioned in the previous section, a

necessary requirement for this coefficients is that the "a_{ii}" coefficients must be strictly smaller than one, while the others can be greater than one. The following image can therefore be seen as the representation in the model of the the matrix **A** of technical coefficients in the previous chapters.



In the procedure `setup` we ask `N_firms` patches to create as many firms. Then we assign the sectors according to the chosen proportion, and we also determine the current needs of the firm, imposing that they operate at scale 1. Because of our constant return of scale assumption, such needs are determined as the product of the scale and the various coefficients of production.

```
ask n-of N_firms patches [set pcolor grey
                          sprout-firms 1 [ set s 1 set active 1]]

ask n-of (N_iron * N_firms)  firms with [sector = 0]
[set sector "iron"
 set iron_need  s * iron_iron
 set coal_need  s * coal_iron
 set wheat_need s * wheat_iron]
```

We then provide the firms with the necessary amount of inputs to start the production

```
ask firms [set iron_have  iron_need
           set coal_have  coal_need
           set wheat_have wheat_need]
```

In the following section we will describe the code used to make the firms produce, look for replacement of inputs, and decide if they can invest.

6.4 Produce, Replace and Invest

The procedure `produce` simply ask the firms to erase the value of the inputs they possess and produce an amount of `output` equal to their scale.

In the following two procedures, `preserve_inputs` and `find_inputs`, on the other hand, firms look for a replacement of the inputs consumed in the previous procedure. In the first procedure, all firms subtract from the output they produced the amount they will need the following period (`*_need`), and they add it to their store (`*_have`):

```
if sector = "iron"  [set output output - iron_need  set iron_have  iron_need]
```

The subsequent search for the remaining inputs is done with a cycle, since we want that all firms perform the task sequentially and we do not want the order to be always the same. We assign to the firm a variable `quest` which will take value one once the research for the inputs is done. The cycle therefore will continue until there are not any more firms with `quest = 0`. Let us consider the search of a firm for the amount of iron it will need the subsequent period: first of all, if our firm belongs to the iron industry, it already possess the input it will need from the previous procedure. Its search is done, then. Otherwise, so if `iron_have < iron_need`, first of all it checks if the total remaining output of the firms producing coal is enough to satisfy its needs, `iron_need`. If this is not the case, it means that the firm will not be able to restart its production in the following period, so the firm is doomed to disappear. If, on the other hand, there is enough iron left in the whole economy, the firm will start a cycle that will keep going until its `iron_have` will be equal to its `iron_need`. In this cycle the firm will select the firms in the iron industries which are not out of stock, and ask them to set a variable `sell` equal to the minimum between the amount that would satisfy its needs (`iron_need - iron_have`) and the remaining output of the industry. In this way, if the firm has coal enough, the iron firm has finished its journey. Otherwise, it will take all the output that the coal firm has available. The process is described by the following code:

```
if iron_have < iron_need
  [if-else sum [output] of firms with [sector = "iron"] >= iron_need
     [while [iron_have < iron_need]
        [ask one-of firms with [sector = "iron" and output != 0]
            [set sell
              (min list ([iron_need] of myself - [iron_have] of myself) output)
            set output output - sell
            ask myself
              [set iron_have (iron_have + [sell] of myself)]]]]
     [set active 0]]
```

After the firm has found the iron it needs, it will follow an analogous procedure to gain its necessities of coal and wheat.

Once that every firm has performed the above task, the firms with `active = 0` are eliminated, and the remaining firms are asked to set `quest` equal to zero again. At the end of this procedure, any "survived" firm has stored whatever it will need in the following period: into its variable `iron_have` the amount of iron (which will be equal to `iron_need`), in `coal_have` the amount of coal and in `wheat_have` the amount of wheat.

The sum of the output of each sector that has not been claimed by anyone is then aggregated in the correspondent surplus global variable, such as in

```
set surplus_iron  sum [output] of firms with [sector = "iron"]
```

The `invest` procedure determine how many firms will increase the scale of their production in the following period. Also in this case we make use of a cycle and of a specific variable, `quest2`. A firm with `quest2 = 0` is asked to check whether there is enough surplus of each commodity to increase its production of one unit. If this is the case, the firm add to its `*_have` variables such amount, subtracting it from the three surpluses, then increase its scale `s` of one, and finally adjourn its `*_need` variables for the subsequent period by adding to its previous needs the necessary amount for producing one unit, that is `*_need / s`. If, on the other hand, there is not enough surplus of even one commodity to sustain an increase in output, the firm will ask all the other firms in its sector to set `quest2 = 1`. In fact, if it cannot find enough of some input, neither it would any other firm using the same method of production (while it may be the case that a firm in an other industry, using different amount of various commodities, could). When all the firms have `quest2 = 1` the cycle stops. Notice that with the described procedure it is possible that a firm increase its scale more than one time in the same period.

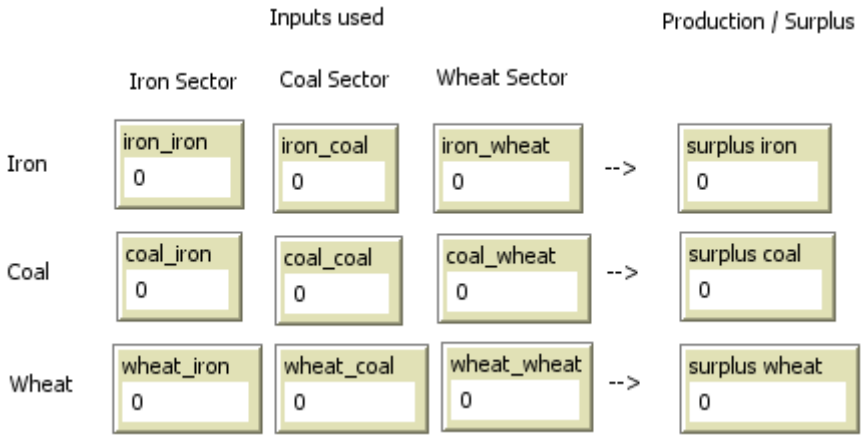
```
while [any? firms with [quest2 = 0]]
  [ask one-of firms with [quest2 = 0]
     [if-else   ((surplus_iron  > (iron_need  / s))
             and (surplus_coal  > (coal_need  / s))
             and (surplus_wheat > (wheat_need / s)))

        [set surplus_iron  surplus_iron  - (iron_need  / s)
         set surplus_coal  surplus_coal  - (coal_need  / s)
         set surplus_wheat surplus_wheat - (wheat_need / s)

          set iron_need  iron_need   + (iron_need / s)
          set iron_have  iron_need
          set coal_need  coal_need   + (coal_need / s)
          set coal_need  coal_have
          set wheat_need wheat_need  + (wheat_need / s)
          set wheat_need wheat_have

          set s s + 1]

         [ask firms with [sector = [sector] of myself]
              [ set quest2 1 ]]]]
```

Finally, the procedure `go` gather the procedures `production, preserve_inputs, find_inputs` and `invest`, representing a whole cycle of the economy.

6.5 Interface

In the interface, beside being able to set some variables as explained above, it is possible to keep track of the evolution of the economy. Through a series of monitors is possible

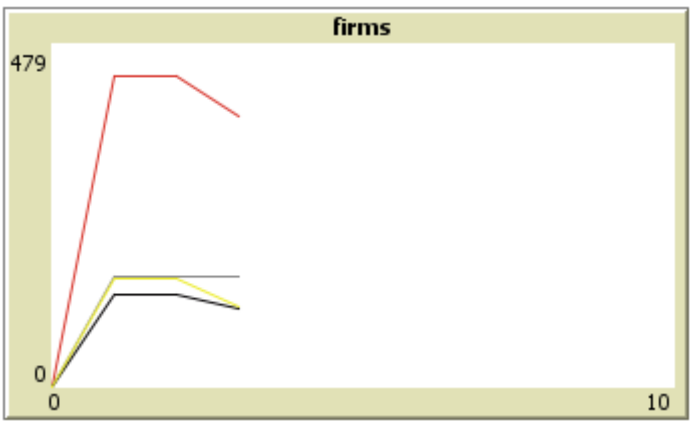
to see the total amount on any commodity used as input in any industry, as well as the quantity produced and the surplus generated.



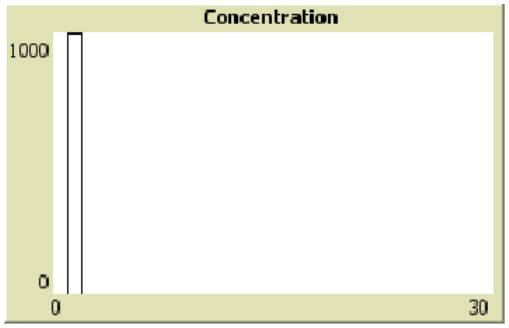
Using the conventions of Chapter 2, the nine monitors on the left represent the first n rows and columns of the Input-Output Table of the economy (in physical terms), while the last column represents either the quantity produced or the surplus (or "final demand"), according to which procedure has been just run. In matrix terms, and still using the conventions of the previous chapters, the monitors represent the amounts

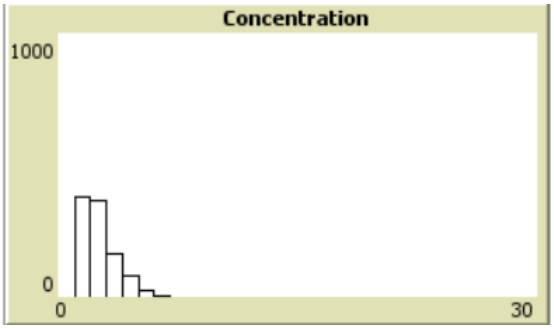
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \tag{6.1}$$

Where Q_1, Q_2 and Q_3 are the scale at which the industries are, overall, operating.
The situation of the economy can be analysed also with the help of two graphs. One keeps track of the number of firms active in any period of time, both totally and for individuals sector. In the following image, for example, we see an economy which had lost some industries in the first research for inputs

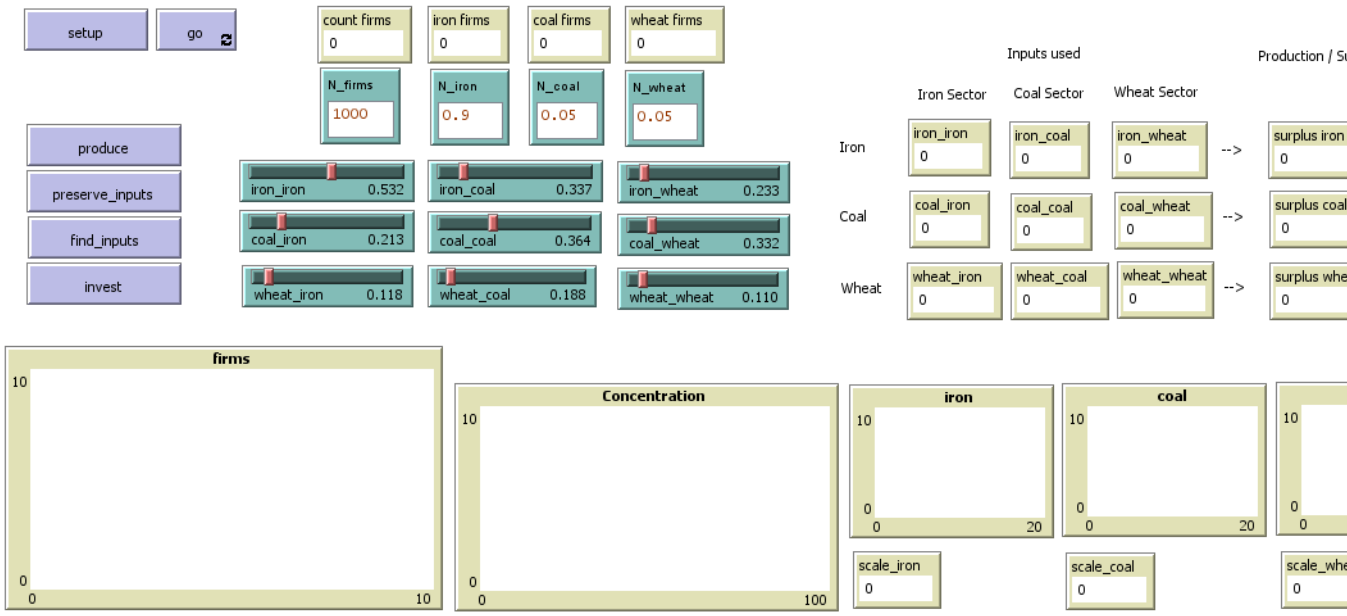


The red line represent the total amount of firms, while the number of firms in the industries of iron, coal and wheat are respectively indicated by colors grey, black and yellow. In the above example we can see that all iron industries were able to find enough inputs, while some firms in the coal and wheat industries could not. This indicates a shortage in the production of iron.
An other useful graph is the one indication the "concentration' in the economy, meaning the amount of firms operating at different levels of scale.





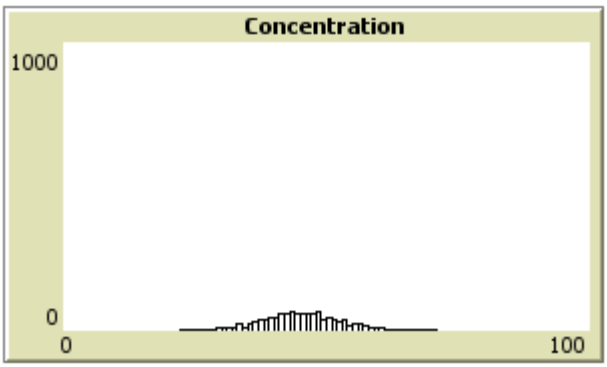
In the left graph all firms (1000 firms) are operating at scale one, while on the right one we can see an economy growing, with many firms operating at a larger scale. Analogous graphs are present for each different sector.
The complete Interface of an economy right after the `setup` procedure appears in this way:



6.6 Evidence

According to the chosen number of firms, initial proportion among industries and chosen coefficients of production, but still influenced by chance, a specific industry may follow different paths:

I) Perpetual growth.
An industry may start to increase the scale of its production and never stop. If the initial proportions are such that, given the chosen coefficients of production, industries are able to produce enough surplus of all three commodities, a virtuous cycle may arise, and the economy may enter in a never-ending growth path. This is possible since we have assumed that there is no natural scarcity of resources. Notice that the fact that an industry grows means that no firm of that industry has died in the `find_inputs` procedure (otherwise there would not be output left enough to increase the scale). The figure below shows an economy in which all industries are growing, with all firms operating at large scale.



II) Growth and stabilization.
An industry may also grow for a while and then stabilize, producing at a fixed scale the same surplus year after year (in this case we can interpret that such surplus as completely consumed). This happen when the relative growth of sectors makes the demand for a certain commodity used by the industry we are considering to completely exhaust the surplus production of such commodity (or, at least, leave less than necessary for the production of one more unit of commodity). At that point, no firm in our industry can invest, and if also the other industries faces such shortage, the economy enters in a stationary state.

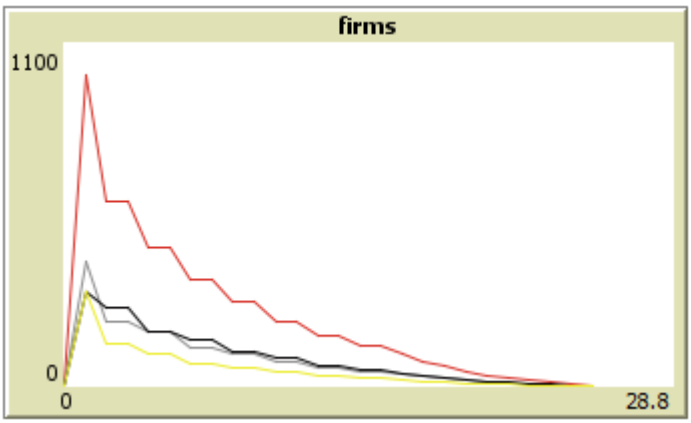
III) Recession and stabilization.
On the other hand, it is also possible that an industry stabilizes at a lower scale of production than the initial one, which means that some firms have died. If the proportions among industries are such that, given the coefficients of production, there is not enough output to replace the means of production of the whole industry then some firms will have to close. If such an elimination will not create scarcity

of the commodity produced by the industry in question (see point VI), then the economy could stabilize (and the industry with it) at a lower level.

IV) Recession and growth.
It is possible, on the other hand, that an industry initially shrinks, and then start growing, entering in either one of cases I and II. This may happen if the industry in question uses for its production an amount of a commodity greater than the amount used by the industry that produces such a commodity. To make an example, consider the case in which some coal-producing firms closes because they are not able to find the necessary amount of iron. If, anyway, the iron technology employ, for unit of production, a smaller amount of iron than the one required by the coal technology, then there is a chance that the iron industry will increase its scale in the following period, producing enough iron to sustain a grown even in the coal industry.[1]

V) Growth and recession.
It is also possible that an industry starts investing but then shrinks its level of production. This may happen if at the initial situation the industry, which uses a relatively low amount of some commodity, has the possibility to grow, but some other industries could not even find the resources to reproduce themselves. In this way the industry in consideration may not be in the condition of reproduce its scale in the next period, given the reduced production of other commodities.

VI) Collapse.
It may happen than the loss of some firms due to the reasons mentioned case IV would start a vicious cycle that could bring to the collapse to each industry, and therefore of the economy itself, as in the following figure.



[1]Of course more complex dynamics may occur: for example the death of some coal firms may release the following period enough resources that would allow the wheat industry to grow, and this in turn could permit to the iron industry to increase its scale. But there are too many possible examples.

In fact, the recession in some industries may create a scarcity of some inputs that could bring other industries to failure, and so on.

Lacking a measure of value we are not able to make a rigorous analysis of the possible dynamics of the economy on the whole. Indeed, apart from the extreme results (all industries grow and all industries shrink), we would not be able to judge a situation in which some industries are boasting and other are reducing their scale.

We can say, anyway, that if an economy is in a downward spiral in which all industries are collapsing, the iterative mechanism which regulate the failure of firms permits the possibility that the economy stabilizes or even start growing in some sector. This because the elimination of some firms may release enough resources for an other industry to stop shrinking, and maybe stabilize or invest.

At the opposite case, anyway, we do not find a symmetrical possibility of reversing the fate of the economy: if all industries are growing, there cannot be no recession. This because the choice whether to increase the scale or not is done on what we could call a "Feasibility Principle": a firm check if there is enough surplus for it to increase its production, and if there is, then the firm takes it. I all industries are growing this means that no firms have shut down in this period, so that in the following period there will be *at least* as much means of production as this period. The iterative removal of the necessary means of production from the common pool of the surpluses imply that the economy will at least be able to satisfy the newly decided scale of production. Moreover, lacking a price system, there cannot be complications deriving from being not able any more to afford what was previously employed. In conclusion, the lack of a price system and the implicit condition of Perfect Information prevent the possibility of a downturn in a growing economy.

The destiny of the economy, as we have seen, depends on the interaction between three factors: the coefficients of production, the initial proportions between industries, and chance. Let us now analyse in greater detail what is the relative influence of each of those:

i) The Matrix of technical coefficients.
As we have said above, the technical coefficient of the "in-house produced input" must be smaller than one, otherwise the economy would collapse for sure, while the other coefficients do not have such requirements. We may wonder, anyway, if there are requirements that have to be satisfied to guarantee the *possibility* of the generation of surplus in all three industries, which, as it follows from the discussion above, is the requirement for the possibility of investment. Indeed there is a mathematical condition for that, which we can recall from the previous chapters. Our problem can be stated mathematically as the search of the sufficient conditions for which the following system has a solution:

$$\begin{cases} (\mathbf{I} - \mathbf{A})\mathbf{q} = \mathbf{y} \\ \mathbf{q} \geq \mathbf{0} \\ \mathbf{y} > \mathbf{0} \end{cases} \tag{6.2}$$

Where $\mathbf{q}$ is the vector containing the *total scales* of production of each industry, and $\mathbf{y}$ contains the total surplus produced by each industry. Now, in Proposition 3.4.1 on Leontief's Open Model we have proved that a condition for which, *for any* non negative $\mathbf{y}$, there exist an unique non-negative solution $\mathbf{q}$ is that the maximum eigenvalue of $\mathbf{A}$ must be strictly smaller than one. Therefore, if such condition is satisfied, it is always possible to find a proportion among industries (intended as proportion among industries' scales) that permits the generation of a positive surplus (actually, of *any* positive surplus). We have therefore guaranteed the possibility of growing. We call an economy which satisfies such condition to be *viable*, as in previous chapters.
Two more considerations on this subject. First of all, if the condition regarding the maximum eigenvalue is not satisfied it does not mean that the possibility is precluded, just it is not guaranteed (the condition is sufficient but not necessary). Secondly, *ceteris paribus*, increasing progressively one coefficient will have the result of eventually destroy the economy: in fact, the requirement of a specific input will, eventually, become greater than the productive maximum capacity given by the actual scale.[2] Decreasing a coefficient (even up to zero), on the other hand, do not have the opposite effect, but the results depends on proportions and chances.

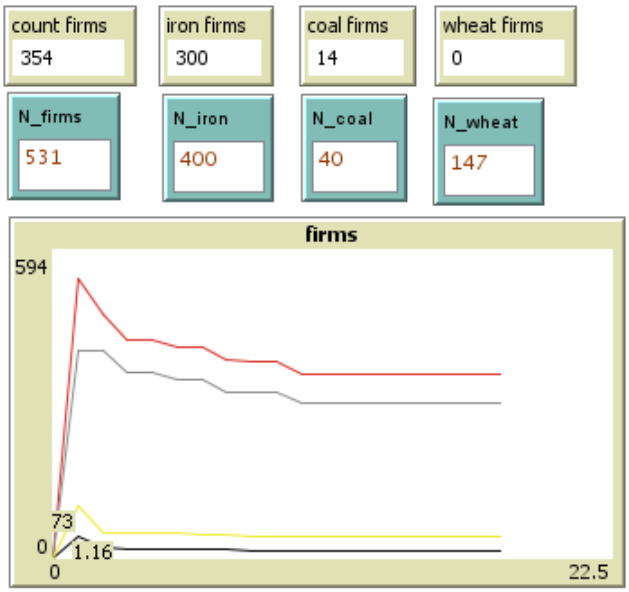
ii) Proportions among total scales.
If the Matrix of technical coefficient can give us the possibility of growth, the actual sustainability of an economy will also depend on the proportions among the total scales of the three industries. It is clear, for example, that if the technology for producing one unit of iron requires two units of coal, we will need the coal industry to be *at least* two times bigger than the iron industry. If the condition of viability is satisfied, there are many possible combinations of strictly positive $\mathbf{q}$ and $\mathbf{y}$. Consider anyway the effects of the gradual increase of the scale of an industry in an economy which generates a surplus in all industries. *Ceteris paribus*, this will prevent the economy from generating a surplus in each other industry sector, since the requirement of the increased industry will be eventually greater to their scale, and therefore on the possibility of growth. At a certain level, it will create actually a deficit for some commodity, and some firms will have to shut down the activity. Analogously, the progressive decrease in the scale of an industry producing a basic commodity (as defined by Sraffa (1960)) will eventually create a deficit in the surplus of such a commodity, with analogous consequences. The results of these "failures" would then depend on chance.

iii) Chances.
The iterative process through which the firms look for inputs do not determine whether an economy will be able to produce a net surplus of all commodities

[2]Notice that the greatest eigenvalue is a direct, continuous function of any coefficient of the matrix (as proved in Proposition 10.2.6), so that increasing progressively a coefficient will eventually kill the sufficient condition of Proposition 3.4.1.

(condition to grow) or not, which is determined by the condition above on technical coefficients and initial proportion among industries. Anyway, if a net surplus in all commodities is created, for certain values of coefficients and proportions may be determinant in deciding which industries will grow, and therefore in which of cases I, II or V each industry will find itself. Indeed the relative grow among scales could determine the exhaustion of the surplus of a specific commodity, and therefore stop the growth at a stationary level.

Analogously, if we consider an initial situation in which the combination of matrix $\mathbf{A}$ and proportion among industries does not permit the complete reproduction of the system, chance may determine if the starting from a situation in which the economy does not produce enough to replicate itself in the subsequent period, chance may be determinant in deciding whether the industries will find themselves in cases III, IV, or VI, so that the economy will recover, stabilize or collapse completely. Indeed, if the condition about the maximum eigenvalue is satisfied, the possibility of generating a non-negative vector of surpluses exist, and whether such proportions among industries would arise in the reiterative removal of firms depends on the random order of firm's research for inputs.
Consider for example an economy with the production coefficients of the subsistence economy described by the Input-Output table 3.1 of Chapter 4. A subsistence economy can be seen as a Leontief Closed Model (see Chapter 4), and Proposition 3.3.1 tells us that there is *only one* relative quantity vector that satisfy this system: in particular, total scales of production will have to respect the proportions $450 : 21 : 60$, or $150 : 7 : 20$. If we observe the evolution such an economy starting from a situation in which we do not have such proportions among total scales, most of the times the economy will progressively grow and collapse. Sometimes, however, it may arise a situation like that one illustrated by the image below, in which the economy is stabilized: this has happened because the number of firms in the three industries, which varies randomly following the iterative process of elimination of firms, has happened to be respectively $300, 14$ and 40, which satisfies the required proportions $150 : 7 : 20$.



We define an economy which satisfy the condition of viability and with the total scales of the various industries in such proportions to guarantee a non-negative surplus for each commodity as in *self-replacing state*, as discussed in previous chapters. In the following discussion we will concentrate our analysis on this kind of economies.

Chapter 7

Model Two: purely physical model with distribution of surplus

7.1 Introduction

Let us now try to develop our model considering the introduction of workers, and therefore of labor as the only scarce factor of production. We will therefore add a new category of agents, the `workers`. We will suppose that firms need labour in order to produce, and that they can hire workers in exchange of a wage. Since we are still considering a purely physical economy, lacking a price system, this wage will be composed of three parts, one for each commodity.

Initially we will consider labor exactly as any other commodity, in a way that can resemble the approach of Leontief's Closed Model (see Chapter 3). This means that employing an unit of labour will cost the firm a fixed amount of each commodity, and such amount will be considered as an addition to the necessary means of production. This is the approach used by Von Neumann, when he considers that the workers will receive only the subsistence wages, which remains constant over time. It is clear that this approach will not have many differences with respect to the one used until now, except for the fact that labor is necessarily scarce, so there will be a limit to the possible accumulation.[1]

After that, we will move away then from the case of a fixed salary in order to investigate the effects of a "surplus" theory of distribution, in which total wages are determined as a share of net product. This is the approach used by Sraffa, and in general is in the tradition of classical economists. As in the discussion before (see section 4.3.1), it may be useful to consider the wage only as the "surplus" wage, that is the wage above the minimum means of subsistence for workers (such subsistence would be included into the "technical" coefficients of production). Such surplus wage would be determined from

[1] We move away, therefore, from the hypothesis of complete absence of scarce resources that had characterized the previous model.

the bargaining between the two social classes, it would therefore have a social determination. After considering the case in which such determination is fixed exogenously, we will consider a simple modelization of a conflictual bargaining, in which the share going to the workers will be a direct function of the employment rate (to symbolize the increased "bargaining power" that the working class would have in an almost saturated labor market).11

We will finally consider the effects on our simple economy of what we may call "incomplete information" in the process of determination of wage, when today's data is not known and the unit wage(s) of tomorrow are determined on the basis of today's surplus.

One final consideration: in the discussion below we will mostly concentrate our analysis on economies in self-replacing state , in which the technical coefficients and initial proportions among industries are such to permit the determination of a strictly positive surplus in all three industries, and therefore the possibility of investment. In particular we will investigate how the accumulation path of those economies that, in the previous model, would have experienced perpetual growth will be affected by the distribution process as we have modelized it.

7.2 New Procedures

In order to modelize the presence of workers we have to make some modification to the procedures described in the previous section, as well as creating new ones.
In the `Setup` procedure we create `N_workers` agents denoted as `workers` (the number of workers can be determined from the interface). The `workers` have two built-in variables: `job`, a dummy variable indicating if the agent is currently employed in the production process, and `w_sector`, indicating, if the worker is employed, in which sector is working.

We have also to specify the amount of workers that any production process need in order to be activated. This can be determined by a counter in the interface, that allows the determination of the global variables `lab_iron, lab_coal` and `lab_wheat`, indicating the amount of labor necessary for the production of one commodity in the various industries. We consider the choice of number of workers to be between 1 (very capital intensive) and 10 (very labor intensive) (of course these values are arbitrarily and easy to modify).

We add also some global variables indicating the wage that will be paid in exchange for a unit of labor. As we have already mentioned, the absence of the determination of value makes us unable to aggregate heterogeneous commodities, so that we will have to specify the amounts of each commodity that is given as wage. Such amounts are indicated by the global variables `w_unit_iron`, `w_unit_coal` and `w_unit_wheat`, and can be either determined endogenously or set exogenously from the interface.

In order to modelize the procedures of hiring, firing and paying workers, we have to include some variables that are proper of each specific firm. As before, we distinguish between the amount of inputs (included labor) currently *needed* by the firms in order to produce at the current scale and the amount of inputs currently *owned* (or, more appropriately, hired). Therefore we define `labor_need` as the number of workers necessary for a firm to operate at scale `s`; `labor_have` as the number of workers currently working for the firm; `iron_need_w, coal_need_w` and `wheat_need_w` are the amounts of respectevely iron, coal and wheat that a firm necessitates to pay current wages; `iron_have_w, coal_have_w` and `wheat_have_w`, on the other hand, are the amounts of the three commodities currently owned by the firms destined to pay wages. The variable `labor_need` is set for the first time in the `setup` procedure, analogously with `iron_need` etc., equal to the multiplication of the "technical" labor coefficient proper of the method of production of the firm and the scale at which the firm is operating. For a firm in the iron industry we will therefore have

```
[set sector "iron"
 set iron_need  s * iron_iron
 set coal_need  s * coal_iron
 set wheat_need s * wheat_iron
 set labor_need s * lab_iron]
```

The amount of `labor_have` is, on the other hand, set in the following way. A firm needing a number `labor_need` of workers to operate, will ask such amount of workers to come to it, set `job` equal to one and set `w_sector` equal to its sector. `Labor_have` will therefore count the number of employed workers that are currently on the same patch of the firm:

```
ask n-of (labor_need) workers with [job = 0]
[move-to myself set job 1 set w_sector [sector] of myself]

set labor_have count workers-here with [job = 1]
```

We now consider the procedure that describe the search of the firms for the inputs necessary to pay current workers. Analogously to the model described in the previous section, lacking a price system, the firms simply go around asking to other firms for the necessaries inputs. The necessaries inputs are simply determined as the product of the different kind of unit wages for the number of workers currently owned by the firms. For example the amount of iron needed for wages by a generic firm is `set iron_need_w  (w_unit_iron  * labor_have)`. Notice that in this way the needed amount of a commodity for wages is equal to the product between the unit wage in that commodity, the technical coefficient specific to the method of production employed, and the scale at which the firm is currently operating. This because `labor_have` is, by construction, equal to `labor_need` (which is, in turn, the product of the coefficient and the scale).

The procedure for the search of inputs to pay wages, `find_inputs_for_wages`, is rather similar to the `find_inputs` procedure of the previous model.
We add, anyway, a refinement to the iterative process described in the previous section. We now give the possibility to the firms unable to find enough wage to pay currently employed workers to diminish the scale of its operation instead of shutting down activities. Therefore, the firms are first asked to check whether there is enough output of a specific commodity left to pay for the wages necessary to the production of *just one unit* of product. Such necessary amount is determined by dividing the current need of the firm for such commodity for the scale it is currently operating, as clear from the discussion above. If this condition is not satisfied, then the firm cannot find the means of production to operate at all, so is set as non active and later removed from the economy. If, on the other hand, there is enough output left for the firm to operate at *the minimum* level of production, then the firm check whether there is enough commodity left to pay all the workers necessary to operate at the *current* level. If this condition is satisfied, then the firm starts a loop analogous to the one described above for the previous model, simply substituting the `*_need` and `*_have` variables with the correspondent `*_need_w` and `*_have_w` variables. If, on the other hand, the remaining commodities are not enough to sustain the current level of scale, the firm activates the procedure `scale`, in which its scale is reduced by one, the workers in excess are fired and the `*_need` variables are updated to the new scale.

```
to scale

 set iron_need_w  iron_need_w  - (w_unit_iron * (labor_need/s))
 set coal_need_w  coal_need_w  - (w_unit_coal * (labor_need/s))
 set wheat_need_w wheat_need_w - (w_unit_wheat * (labor_need/s))

 set iron_need  iron_need   - (iron_need /s)
 set coal_need  coal_need   - (coal_need /s)
 set wheat_need wheat_need  - (wheat_need/s)

 ask n-of (labor_need/s) workers-here with [job = 1] [set job 0 set w_sector 0]
 set labor_have count workers-here with [job = 1]
 set labor_need labor_need - (labor_need/s)

 set s s - 1

end
```

Notice that, for every scale, dividing a `*_need` or a `*_need_w` variable for the current scale give us the necessary commodities for the production of just one unit of product. Subtracting this from the previous values of the variables gives us the updated value. The firms then checks if the remainders of the commodities are enough to satisfy its needs at this reduced scale. If they are, the cycle for obtaining them starts, otherwise

the firm lower its scale again, and so on.

The complete procedure for the search of the amount of iron necessary to pay wages is therefore

```
if iron_have_w < iron_need_w and active = 1
  [ if-else sum [output] of firms with [sector = "iron"] >= (iron_need_w / s)
     [ if-else sum [output] of firms with [sector = "iron"] >= iron_need_w
        [ while [iron_have_w < iron_need_w]
            [ask one-of firms with [sector = "iron" and output != 0]
               [set sell
                 (min list ([iron_need_w] of myself-[iron_have_w] of myself) output)
                set output output - sell
                ask myself
                  [set iron_have_w (iron_have_w + [sell] of myself)]]]]
        [scale
         set quest 0]]
     [set active 0
      ask workers-here with [job = 1]
          [set job 0 set w_sector 0]] ]
```

For the sake of coherence, also the `find_inputs` procedure is modified in a similar fashion, in order to permit a reduction of scale.

Finally, we create three procedures for simple actions:

The `pay_wages` procedures represent the act of paying wages, so that the commodities stored into the `*_have_w` variables are set equal to zero.

The `surplus` procedure determine the amount of commodities left after that the means of production necessary to restore the production process are removed. In the previous model this operation was done inside the procedure `find_inputs` but we have isolated it for need of grater flexibility in setting the order of economic actions, as we will see in the following subsections.

Analogously, the procedure `INV` determines the amount of commodities left after that both physical means of production and wages are removed from the output, that is that parts of production that are available for investment, included in the global variables `INV_iron, INV_coal` and `INV_wheat`.

As mentioned before, we will now present the results of three different models obtained by the reshuffle of the procedures described above, which correspond to different hypothesis regarding the distributive process

7.3 "Workers like horses"

In a first, intermediate step we consider the workers exactly as any other mean of production. The wages are fixed and need to be paid in advance, before that the production process takes place. The addition of the workers, given this hypothesis, is equivalent to a modification of the coefficients of production in the previous model. Indeed, consider the following example. Say that the amount of workers needed to produce one unit of iron is equal to `lab_iron`. If the (fixed) unit wages for the three commodities are equal respectively to `w_unit_iron, w_unit_coal` and `w_unit_wheat` and the coefficients of production of the iron industry are equal to `iron_iron', coal_iron'` and `wheat_iron'`, then the functioning of such economy will be equal to the one of an other economy without workers but with coefficients of production equal to

```
iron_iron'' = iron_iron' + lab_iron * w_unit_iron
coal_iron'' = coal_iron' + lab_iron * w_unit_coal
wheat_iron'' = wheat_iron' + lab_iron * w_unit_wheat
```

Of course the expenses for workers, since they have to be paid at the beginning of the production process, will have to be kept in consideration when the investment decisions are made. We therefore transform the `invest` procedure into a similar `invest_horses` procedure where also those costs are kept into consideration. Consider, also, that we have to consider, in order for the investment to be possible, also the further condition that the number of unemployed workers is sufficient to sustain an increase in production. The modified procedure therefore appears as:

```
while [any? firms with [quest2 = 0]]
 [ask one-of firms with [quest2 = 0]
   [if-else INV_iron  > (iron_need/s  + (labor_need * w_unit_iron)/s)
      and INV_coal  > (coal_need/s  + (labor_need * w_unit_coal)/s)
      and INV_wheat > (wheat_need/s + (labor_need * w_unit_wheat/s)
      and (count workers with [job = 0] >= (labor_need/s))
        [set INV_iron  INV_iron  -(iron_need/s  + (labor_need * w_unit_iron)/s)
         set INV_coal  INV_coal  -(coal_need/s  + (labor_need * w_unit_coal)/s)
         set INV_wheat INV_wheat -(wheat_need/s + (labor_need * w_unit_wheat)/s)

         set iron_need iron_need   + (iron_need  / s)
         set coal_need coal_need   + (coal_need  / s)
         set wheat_need wheat_need + (wheat_need / s)

         ask n-of (labor_need / s) workers with [job = 0]
            [move-to myself set job 1 set w_sector [sector] of myself]
         set labor_have count workers-here with [job = 1]
         set labor_need labor_have

         set s s + 1]
```

```
        [ask firms with [sector = [sector] of myself] [ set quest2 1 ]]]]
```

The functioning of the economy based on these assumptions is therefore described by the `go_horses` procedure, which gather the described procedures in the following order:

```
to go_horses

  pay_wages
  produce
  find_inputs
  find_inputs_for_wages
  surplus
  INV
  invest_horses

end
```

Notice that, of course, with this order the amount of surplus and of investment fund will be equal for every commodity every period.

7.3.1 Evidence

As we have said, there is little difference between this intermediate-step model and the one that we have analysed before. Of course the considerations about the possible evolution of the scale of an industry still holds, as well as the considerations regarding the role of the matrix of technical coefficients,[2] proportions among industries and chances. The main difference between the two is the presence of a scarce resource, human labor. This, when we consider an economy in self replacing state that otherwise would have grown forever, set an upper bound to the possible level of scale achievable in every industry.

This upper bound allow us to make some considerations on the role of chances in the evolution of the economy. Despite, as we have discussed in the previous section, chances may play a role in the possibility of growth or recession, and even turning the fate of an economy, if we observe the evolution of the same economy (in self replacing state) many times (meaning that coefficients and proportions are not modified), we can see that the attained level of scale for the industries does not have much variability. This means that, despite some variability given by the iterative process or research of inputs, the maximum level of scale obtainable by an industry (and consequently by the whole economy) is approximately determined by the dimension of the available labor force, a

[2] Notice that we would need to consider the "implicit' matrix of technical coefficient resulting from the consideration inside the "technical" coefficients of the labor needs, as described in the previous section.

result which resembles the consideration we have made on Leontief's Closed Model in Chapter 4.

7.4 Wage as proportion of Net Product

Let us now move away from the hypothesis of advanced salaries and consider the case, analysed by Sraffa (1960), of wages paid at the end of the production process as a share of the net product. As mentioned in the Introduction, we consider this wage to be only the part of wage that exceed the minimum subsistence level for the workers to survive, while we consider the subsistence part to be included inside the technical coefficients as explained before. We can initially presume that such proportion is determined outside the production process, as a result of some form of bargaining between workers and firms. We can then set it exogenously with the appropriate slider in the interface

We have then to create an appropriate procedure to set the salary, that, given our assumptions, will differ from one period to an other, following the evolution of the surpluses of the various commodities.
We define then the global variables `w_share_iron, w_share_coal` and `w_share_wheat` the part of the surpluses of the various commodities that will determine the "age fund" of the economy. The unit salary will be then determined by equally dividing such shares among the working population. These operations are gathered in the procedure `set_wage` as follows

```
to set_wage

  set w distribution
  set w_share_iron  w * (surplus_iron)
  set w_share_coal  w * (surplus_coal)
  set w_share_wheat w * (surplus_wheat)
  if count workers with [job = 1] != 0
  [set w_unit_iron (w_share_iron  / count workers with [job = 1])
   set w_unit_coal (w_share_coal  / count workers with [job = 1])
   set w_unit_wheat(w_share_wheat / count workers with [job = 1])]

end
```

Given these assumptions, the functioning of the economy is described by the procedure `go_distribution`, that gathers the various procedures in the following order:

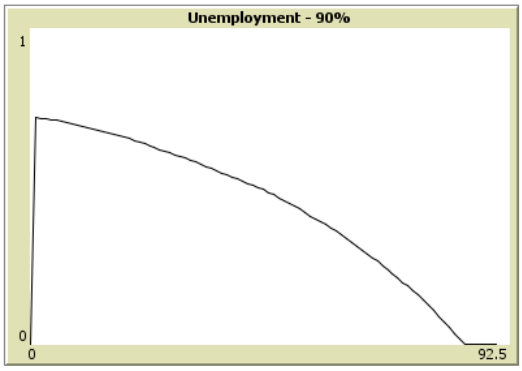
```
to go_distribution

  produce
```

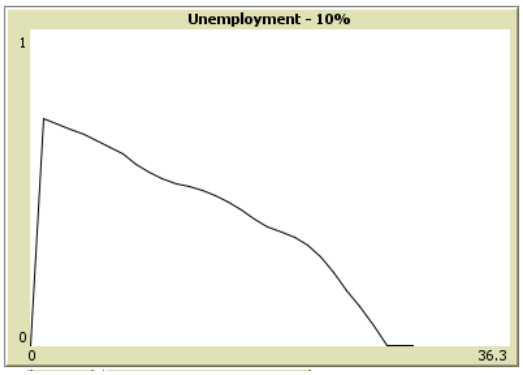
```
  find_inputs
  surplus
  set_wage
  find_inputs_for_wages
  pay_wages
  INV
  invest

end
```

Notice that, differently from the "workers like horses" model, the salaries are paid *after* the production process, according to the values determined in the `set_wage` procedure on the basis of the surpluses determined in the procedure `surplus` as the outputs remaining once that *only* the physical means of production are subtracted. We have therefore a distinction between the `Surplus_*` variables and the `INV_*` variables, and therefore a distinction between that part of the net product that is employed in the increase of the total means of production of the economy and that part that is consumed (the part that goes to wages).
Notice also that the `invest` procedure is the same that we have used in the model without distribution, since now in order to start the production the firms do not have to pay the wages in advance any more, and so have to worry only of the physical means of production.

7.4.1 Evidence

As for the model "workers like horses", the results derived from the first model without distribution still hold for this one, since it consists merely in "burning" part of the surplus, otherwise determined by the same interaction between coefficients of production, proportion among industries and chance.
Let us now consider the consequences on the growth path of an economy in self replacing state. Not surprisingly, the existence of a competitive distribution over the surplus reduces the speed of the accumulation, and this effect is accentuated the greater it is the share going to wages. This is a straightforward result from traditional theory of capital accumulation, since reducing the share of the surplus that is employed in increasing the means of production means reducing the growth rate of such means, and consequently the growth rate of the economy as a the whole.
In the following graphs we can observe the evolution of the unemployment rate in two economies (in self replacing state) that differ only from the fact that in the first one the share of the surplus going to wages is the 90% while in the second one only the 10%.





On the y-axes we have the unemployment rate of the economy, that in both cases goes to zero. On the x-axes, on the other hand, we count the time. As we can see, in the second case the economy employs significantly less time (slightly more than one third) to attain full employment, which determines the end of the possibility of growth. Of course the exact moment of the achievement of full employment will change randomly according to the iterative process of investment, nevertheless, on average, the first economy will employ significantly more time than the second one.
It is worth to notice that even if the amount of surplus going to wages will affect the moment at which the greatest scale of production will be reached (correspondent to exhaustion of labor force), but not, on average, the *level* of such scale, which will be, as before, de facto determined by the number of workers present in the economy. This means that a distribution more favourable to workers will slow down the pace of accumulation, but will not result in a lower output in the "stationary state".

7.4.2 Periodical bargaining

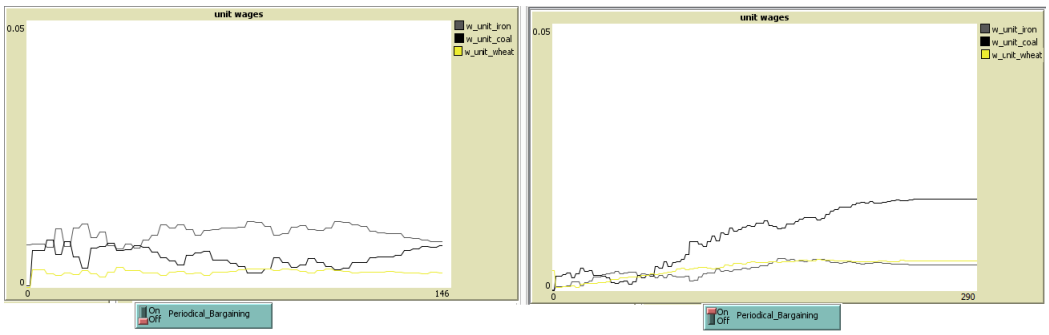
Since now we have assumed that the share of the surplus going to the workers has been bargained once for all before the beginning of our investigation, i.e. that it is exogenously fixed. It is rather simple to relax this assumption in order to represent a periodical redefinition of the bargaining power of the two breeds of agents, workers and firms. We limit our investigation to a very simple case, in which the share going to the workers is directly related (actually, identical) to the employment rate of the population. This represent the well known fact that, in case of high unemployment, the firms are able to impose a lower salary, taking advantage of the excess supply of labor. On the other hand, on a saturated labor market, the bargaining power of the workers is very high and they can obtain higher wages.
This process of periodical bargaining is activated from the interface through a dedicated switch. We add therefore to the procedure `set_wages` the following line of code

```
if Periodical_Bargaining
[set Distribution (count workers with [job = 1] / count workers)]
```

The effects on this modification to the path of an economy are not surprising. Coherently with our analysis above, the progressive increase of the share destined to the workers will

progressively slow down the accumulation, but never prevent it.

It may be interesting to consider the evolution of unit wages given the functioning of our distributive process, both in the case of exogenous distribution and in the one of periodical bargaining. In both cases the amount of commodity per capita will depend on three forces: the percentage of the surplus going to the workers; the dynamic of the surplus; and the increase in the labor force.
In the first case the percentage of product going to the workers is fixed, so the unit wage(s) will simply follow the dynamic of surplus per capita, which depend strongly on chance. Anyway, repeating the analysis many times for an economy initially in self replacing state, it appears that the surplus per capita (and therefore the unit wage) has the tendency of staying rather stable over time. We can see an example of the dynamic of the unit wages in the left figure below
Also in the case of periodical bargaining the dynamic of unit wage depend on the iterative process of investment and it is therefore rather random, but of course the tendency of appropriating of a grater piece of the cake as long as unemployment falls makes that basically we will witness an increase in the unit wage. We can see an example of that in the figure on the right below



7.5 Incomplete Information

We consider now the case in which the process of bargaining between the two social classes is not based on perfect information about the amount produced and employed this year. The only information available in one period are the quantities composing the surplus of the previous period, and so the wage share are set on the this basis. The consequence of this assumption is the positioning of the `set_wage` procedure at the beginning of the list, as in the following procedure `go_rigidities`.

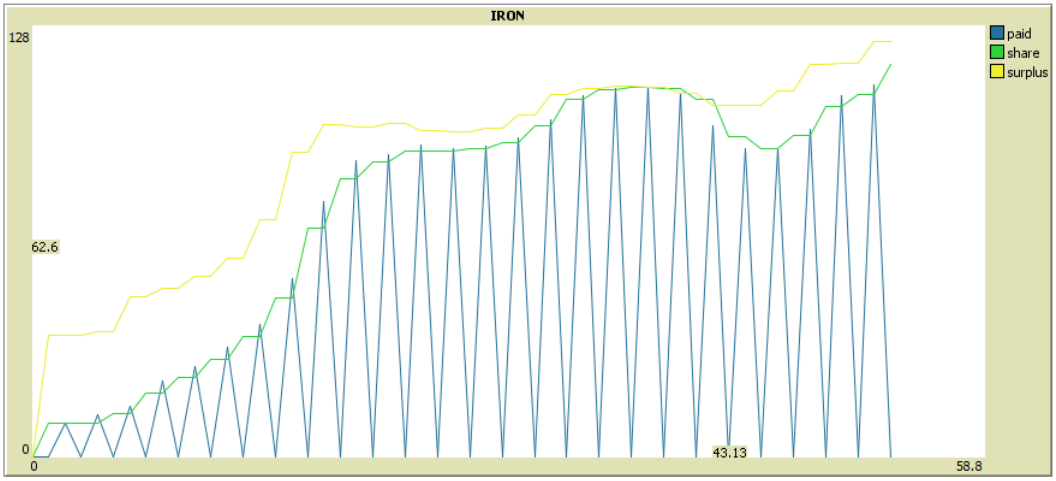
```
to go_rigidities

  set_wage
  produce
  find_inputs
```

```
  surplus
  find_inputs_for_wages
  pay_wages
  INV
  invest

end
```

This assumption can be the source of serious disturbances in the accumulation path of an economy. In fact it creates a difference between the share wage computed on the basis of today's surplus and the amount of resources that will have to be dedicated to the payment of wages tomorrow, and that will be extracted from *tomorrow's* surplus. While in the previous model there could not be imbalances in the payment of wages, since the wage share was computed on the basis of the actual commodities produced, it is not guaranteed at all that tomorrow's surplus will be enough to sustain all the due payments. The sustainability of the values set will depend on the evolution of the surpluses and of the working population (and consequently on the factor affecting these variables).

Consider, as an example, the payment of the part of wages due in iron in a growing economy. Say that today's surplus of iron is 100, that 10 workers are employed in the whole economy. Assume, for simplicity, that teh `Distribution` variable is set equal to one. The `w_unit_iron` set for tomorrow will therefore be 10. Now, the sustainability of this value will depend on tomorrow's surplus of iron and tomorrow's increase in the global working population. If, say, the surplus will increase to 120 and the working population to 11, then there is no problem: 110 units of iron will be paid to workers, and there will also be 10 left for investments. If, on the other hand, the working population increased to 13, there would be an imbalance: 130 units of iron would be due, but only 120 would be available, so some firm will have to shut down (or rescale its production).

In the following graph we can see the evolution of three variable in an economy initially in self replacing state with periodical bargaining : the surplus of iron, the yellow line; the share that yesterday wad assigned to wages (in iron),on the basis of yesterday's surplus, the green line; and the wages (in iron) actually paid today, the blue triangles.



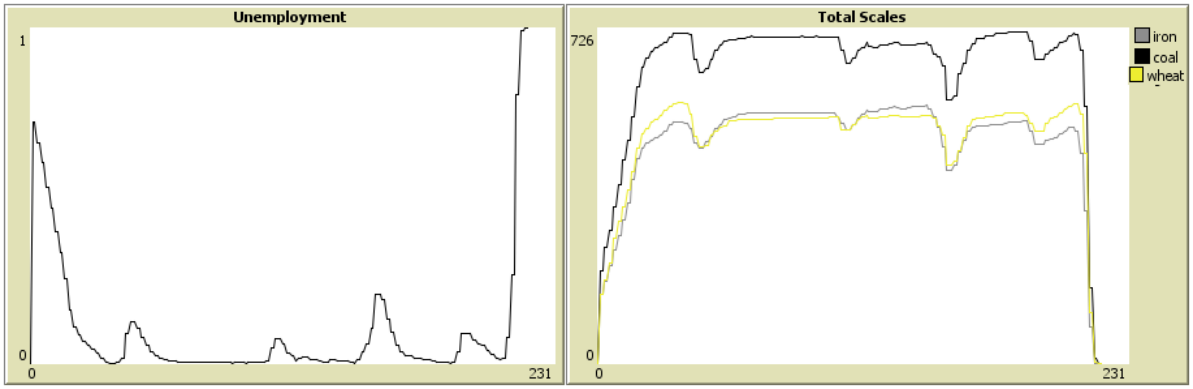
In the first part of the graph, the economy is growing and there are a lot of unemployed. Therefore the share of surplus going to wages is relatively low, around 40%. We can see that the blue triangles (the paid wages) always "perforate" the green line (the "predicted" wages). This is because the economy is absorbing many workers, and therefore the unit wages have to be paid to a greater number of people than the one predicted the previous period. This reduces a little the amount of surplus available for investment (the difference between the yellow line and the tip of the blue triangle) with respect to the case of perfect information, but this does not create particular problems. The problem arise when the unemployment is close to zero, so that the green and yellow line are very close to each other. In this case, as in the central part of the graph, it may well happen that a relative, temporary shortage in iron reduces the surplus at such point that the amount that should be paid exceed the available surplus. We can see this happen when the tip of the blue triangle touches the yellow line of the surplus.[3] This can have destructive consequences for an economy: indeed some firms will die (or lower the scale) in consequence of this, and this may create relative scarcities in the economy that may cause other firms to die, and so initiating a recession. We can see the effects of a recession in the right part of the graph: the surplus is falling, and so are the green line and the blue triangles.

It is worth to point out the countercyclical effect of this of the functioning of the economy: When the economy is growing, as we have seen, the amount of resources available for investment (and therefore growth) is reduced with respect to the perfect information case; on the other hand, when the economy is shrinking, many workers are expelled by the labor force, and this makes the paid wages (blue triangles) to be actually lower than predicted wages, which gives breath to investments and may permit a recovery.

Now, because of the very simple features of our economy, it is almost certain that economies initially in self replacing state with periodical bargaining will sooner or later enter in a series of recessions, and eventually collapse completely.
In the following graphs we can see the effects of such dynamics to the employment and

[3] Notice that the blue triangle represent the wages *actually* paid, so they can never exceed the surplus.

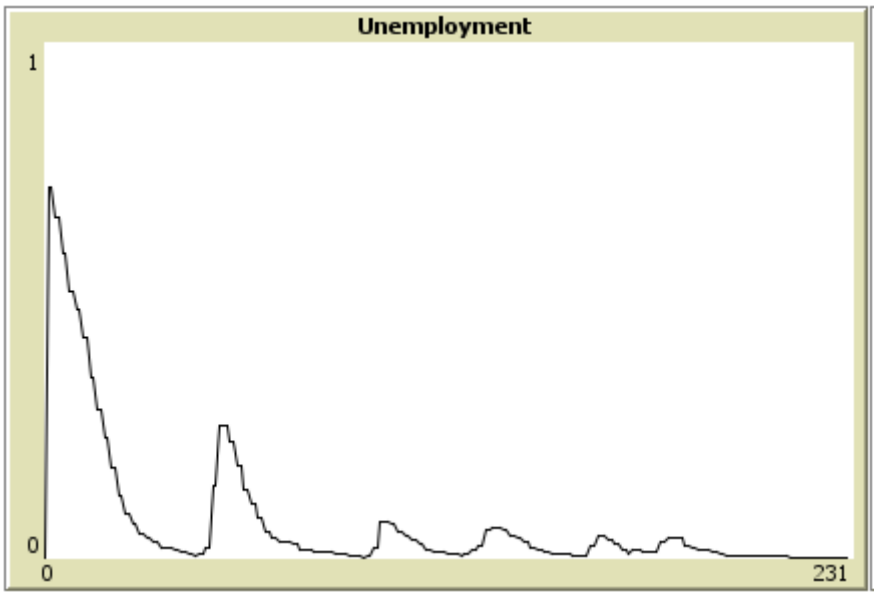
to the total scale of production of the three industries.



This is not a certain result, but it is possible (actually probable) because of three assumptions: the absence of an upper limit to the share of surplus going to the workers; the lack of prudence of firms that invest without being sure that they will be able to find the appropriate resources to pay wages; the fact that the number of agents is fixed, and can only diminish. In fact, the economy enters in a recession because the share of surplus going to the workers is so close to one that an unbalance occur, and during such unbalances some firms shut down the activity. The survived firms start then to grow again, until the following recession, when some other die, and so on until we have very few firms operating at a very large scale, so that the death of even one firm create an enormous unbalance.

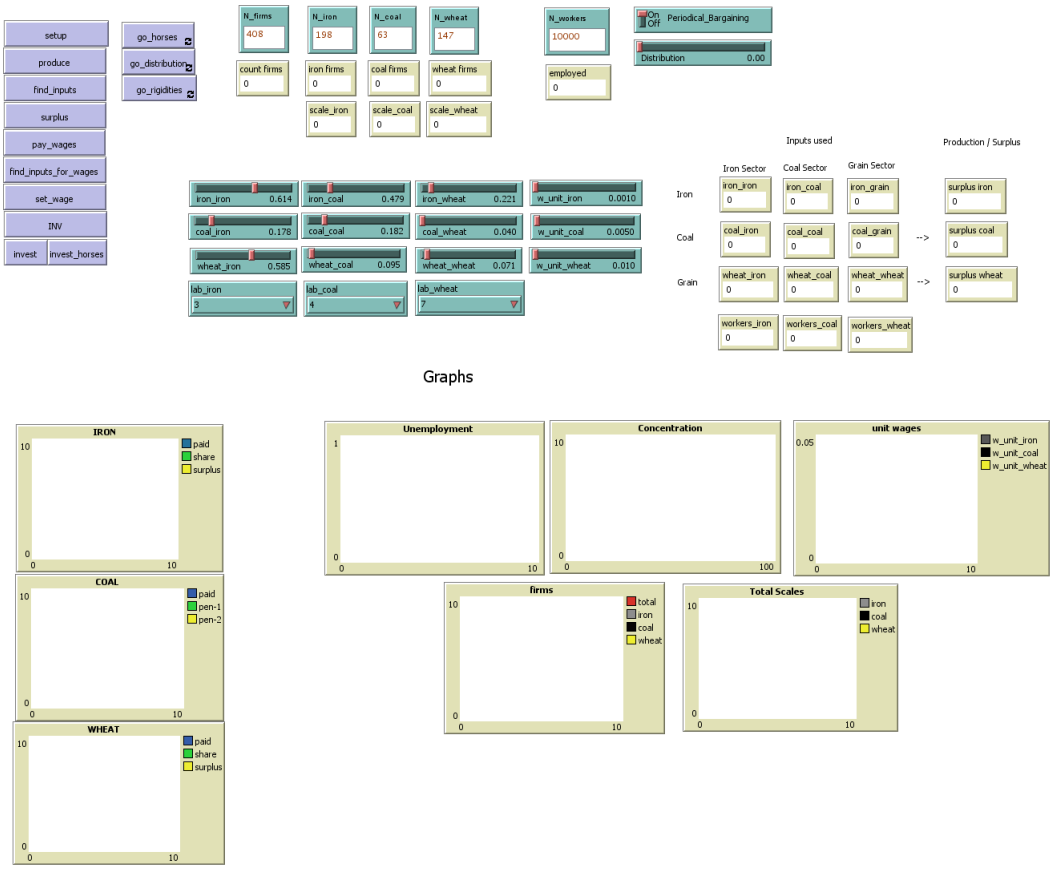
There are many ways of dealing with this unrealistic result. We could, for example, allow the random generation of new firms, which would be able to take advantage of the increased room for investment generated by a sudden unemployment, and of workers, which would keep the unemployment level up and reduce the risk of recession. Or we could set an upper bound to the share of surplus going to workers. We will not follow these roads and leave it to further developments.

An easy way to strongly reduce the probability of a collapse is to set a more "prudent" behaviour of the firms: not knowing the exact amount of commodities they will have to give as salaries in the following period, they invest only up to the level such that they can afford new means of production *and* the new salaries. This is exactly hat firms were doing in the `invest_horses` procedure described above. Substituting such procedure to the `invest` one, the gravity of the recessions is strongly reduced, and the average development of an economy become as the one shown below, with a series of recession of decreasing intensity until a stationary state at a level of almost full employment.


Unemployment
1
0
0
231

7.6 Interface

The Interface, with the addiction of the procedures, inputs and graphs described above results like that



Chapter 8

Model Three: Sraffa's Prices

Let us now move away from a purely physical economy and start to consider the determination of prices.
This is, without any doubt, one of the most fundamental object of inquiry for the economic science. In the previous chapters we have analysed the considerations relative to the determination of prices by three major thinkers of the 20th century, Wassily Leontief, Piero Sraffa and John Von Neumann. In the following section, we will follow an approach that tries to adopt their point of view, that, as we have analysed, relates the determination of prices with the structural necessities of a system rather than the natural scarcity of a resource.
Considering in particular Sraffa's analysis, therefore, we will not set prices through a "bottom up" approach, but as technical and social necessities for a system which has to reproduce itself and pay an uniform rate of profit across industries.
We will then be able to analyse the evolution of the main macro-economic variables in an aggregate way.

8.1 Computation of prices

In order to have Sraffian prices we have to recall their characteristics as outlined in Chapter 4. We have in particular to remember that the starting point of Sraffa's analysis are the *observed* quantities produced and employed in the production process of an economy in self-replacing state. An economy is in self-replacing state, as defined in the previous chapter, if produces, for each commodity, an amount greater than the amount it employs in the production process, i.e. generates a positive surplus in any industry. Sraffian prices, indeed, defined as the necessary prices in order to restore original conditions of production and pay an uniform rate of profit, would not make sense in an economy which is physically incapable of replicate itself. First of all, therefore, we have to create a procedure that allows us to generate an economy in self replacing state.
Recalling the discussion of previous chapters, we have all the elements to do so. Creating an economy in self-replacing state means imposing conditions on both the technical coefficients (the economy must be *viable*, as defined in section 6) and the relative proportions

among the quantities produced by the different industries. As we have already noticed in section 6, a *sufficient* condition for a matrix of technical coefficients to be able to generate a positive surplus is

$$\lambda_{max} < 1 \tag{8.1}$$

Where λ_{max} is the greatest eigenvalue of the matrix. We remember also from subsection 4.3.1 that this is also a necessary condition for the maximum rate of profit that guarantees the non-negativity of the price vector to be greater than zero, and therefore economically meaningful. Remember, in fact, that the economically meaningful maximum rate of profit Π is defined as

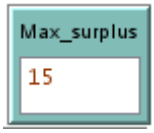
$$\Pi = \frac{1}{\lambda_{max}} - 1 \tag{8.2}$$

The procedure `gen_viable_economy` consists in a loop which generates random values for the production coefficients `iron_iron, coal_iron,...,wheat_wheat` and order such values into a matrix, until the maximum eigenvalue of such a matrix is not smaller than the unit. In order to work with matrices in Netlogo, we have to make use of the extension `matrix`. The process to insert variables into a matrix in Netlogo is shown in the following lines of code[1]

```
let A matrix:from-row-list [[ 0 0 0] [ 0 0 0] [ 0 0 0]]
matrix:set A 0 0 iron_iron
matrix:set A 1 0 coal_iron
matrix:set A 2 0 wheat_iron
matrix:set A 0 1 iron_coal
matrix:set A 1 1 coal_coal
matrix:set A 2 1 wheat_coal
matrix:set A 0 2 iron_wheat
matrix:set A 1 2 coal_wheat
matrix:set A 2 2 wheat_wheat
```

As we have considered in section 6, the condition of viability is not a sufficient condition for an economy to be in self-replacing state, since the production of a surplus in each industry depends also on the relative proportions among the scales of production of the three industries. In order to set such proportions we make use of the result of Proposition 3.4.1, according to which if the condition of viability is satisfied there always exist a non-negative set of scales of production that is able to generate any vector of non-negative surpluses. The procedure `set_proportions` makes use of such result, generating random vector of surpluses and computing the appropriate scales to generate such surpluses. The global variable `Max_surplus` determine the maximum level of surplus that can be generated in each industry, and can be set from the interface.

[1] From now on we will omit to report the code for the construction of matrices since it would be redundant.



Then we generate a random vector in which each element consists in the surplus to be generated in on e industry.

```
let Y matrix:from-row-list [[0] [0] [0]]
  matrix:set Y 0 0 random-float Max_surplus
  matrix:set Y 1 0 random-float Max_surplus
  matrix:set Y 2 0 random-float Max_surplus
```

Since at the beginning of our analysis we consider that each firm produces one unit of commodity, the total production of an industry (the scale at which the industry is operating) is also the total number of firms in that industry. Therefore we set the variables `N_iron, N_coal` and `N_wheat` by solving the system of equations

$$(\mathbf{I} - \mathbf{A})\mathbf{q} = \mathbf{y} \tag{8.3}$$

Where $\mathbf{A}$ is the matrix of technical coefficients, $\mathbf{y}$ the vector of surpluses and $\mathbf{q}$ the vector of scales (i.e. number of firms in each sector).
With the procedures `gen_viable_economy` and `set_proportions` we can then generate random economies in self-replacing state. We can then move on to the determination of prices.

As we have discussed in Chapter 4, in order to solve Sraffa's equation we have to determine exogenously one of the distributive variables, which allows for a bargaining setting of the same. Following Sraffa (1960) and our analysis above, we decide to set the rate of profit as our exogenous variable.

First of all we set the maximum rate of profit `R` from the highest eigenvalue of the matrix of technical coefficients $\mathbf{A}$, as shown in the following line of code[2]

```
set lambda max matrix:real-eigenvalues A
set R ((1 / lambda) - 1)
```

The economically meaningful rate of profit will assume values between zero and `R`, so we have created the following slider in the interface that takes value between zero and one, and that, multiplied by the maximum rate of profit, determines the actual rate of profit `p`.

[2]From expression 4.65.

Let us now determine the prices and wage as the solution of Sraffa's equations. Denoting or variables as `p_i, p_c, p_g` and `wage`, the values depend on the technical coefficients and on the profit `p`, and are obtained solving the following system.

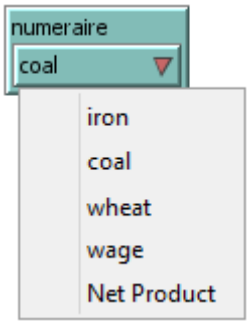
$$\begin{cases} \texttt{(p\_i * iron\_iron + p\_c * coal\_iron + p\_g * wheat\_iron) * (1 + p) + L\_iron * wage = p\_i} \\ \texttt{(p\_c * iron\_coal + p\_c * coal\_coal + p\_g * wheat\_coal) * (1 + p) + L\_coal * wage = p\_c} \\ \texttt{(p\_i * iron\_wheat + p\_c * coal\_wheat + p\_g * wheat\_wheat) * (1 + p) + L\_wheat * wage = p\_g} \end{cases}$$

to which we have to add a fourth equation determining the *numeraire*, which can be either one of the commodities or the value of the Net Product. The variables `L_*` indicates the amount of labor used in the industry `_*` to produce one unit of a commodity.

We solve the system in the procedure `set_prices`, in which we make use of the following reformulation in matrix form of the system above

$$\begin{bmatrix} \texttt{iron\_iron (1 + p) - 1} & \texttt{coal\_iron (1 + p)} & \texttt{wheat\_iron (1 + p)} & \texttt{L\_iron} \\ \texttt{iron\_coal (1 + p)} & \texttt{coal\_coal (1 + p) - 1} & \texttt{wheat\_coal (1 + p)} & \texttt{L\_coal} \\ \texttt{iron\_wheat (1 + p)} & \texttt{coal\_wheat (1 + p)} & \texttt{wheat\_wheat (1 + p) - 1} & \texttt{L\_wheat} \\ \texttt{*} & \texttt{*} & \texttt{*} & \texttt{*} \end{bmatrix} \begin{bmatrix} \texttt{p\_i} \\ \texttt{p\_c} \\ \texttt{p\_g} \\ \texttt{wage} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where the last row of the matrix of coefficients will vary according to the choice of the *numeraire*[3], which can be chosen from the interface.



The procedures, therefore, solve the system and set the values of variables `p_i, p_c, p_g` and `wage`.

8.1.1 Relationship between prices and the rate of profit

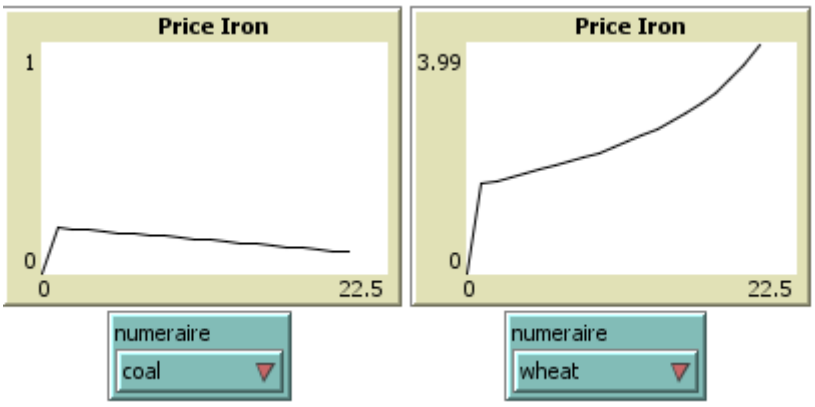
With the help of our simulation we can have a more intuitive perception of the analytical results regarding the relationship between the price of a commodity and the general rate of profits, as discussed in section 4.3.2. As we have said, such relationship is quite complex, and depends on the capital intensity of the commodity whose price is considered and of the commodity used as *numeraire* (what we have called *capital intensity effect*), but also on the complex effect that the different distribution has on the prices of all other

[3] for example its elements will be $1, 0, 0, 0$ if the chosen *numeraire* is iron, $0, 1, 0, 0$ if it is coal, etc.

commodities in the economy.
The procedure `evolution_prices` allows us to see, for any generated economy in self-replacing state, the different prices for many possible values of the interest rate. In particular, the procedure starts by setting `profit` equal to zero, and then it starts a loop which computes the prices correspondent to increasing values of the rate of profit, until it maximum value `R`.

```
to evolution_prices
let c 20
set profit 0
set cycle 0
while [cycle < c]
      [set p profit * R
       set_prices
       update-plots
       set profit profit + (1 / c)
       set cycle cycle + 1]
end
```

We can follow in the Interface the different values of the price of the various commodities on a specific graph. This procedure allows us also to underline the importance of the choice of the commodity used as *numeraire* in the sign of the variation. Consider the following example, in which we consider the different values assumed by the price of wheat, `p_g` through the procedure `evolution_prices` (i.e. for 20 increasing values of `p` from 0 to 1). In both graphics below the variation of the exogenous `p` is the same, but in the image of the left the *numeraire* commodity is coal while in the image of the right is wheat.



As we can see, in the first case we have a decrease in price, while in the second case a distinct increase. Clearly this evolution does not reflect the method of production of the commodity iron, since they are the same in both cases, but merely the relationship with the method of production of the commodity used as *numeraire*. As we have already

mentioned, this lack of transparency was one of the reason for the building of the Standard Commodity, a *numeraire* which does not change value following a change in the distribution.

8.2 Growing economy

Let us now analyse the evolution of of an economy in self-replacing state in which prices are determined in the way described above. We will start by making some strong assumptions regarding the techniques of production and the choice of investment.

First of all, we will assume that the workers will spend everything they receive as wages in consumption. On the other hand, firms will invest as much as they can, meaning as long as they can find the necessary commodities and pay for them, and consume whatever left. Notice that firms have to face an additional constraint in their investment decision, with respect to the ones in the purely physical models. Indeed in that models we assumed that firms could just take whatever they needed for free, while now they are also constrained by the value of their production left once that they have recovered the necessary means to work at their current scale of production and paid the wages to workers. We still assume that, given these constraints, firms will invest as much as they can, which make sense given the assumption made in the following paragraph. Notice, moreover, that the previous assumptions regarding the spending of agents imply that everything is produced will be actually bought.

As in the previous models, we will assume Constant Return to Scale. We have already mentioned that Sraffa does not make such an assumption, nevertheless it does not exclude it neither, since its analysis starts "after the harvest" and he does not consider the actual methods of production or the issue of accumulation. The assumption of constant return to scale has important consequences that facilitates the dynamic of the model. Firs of all, thanks to that assumption, the prices and wage that guarantee that *an industry as a whole* is able to replicate itself and pay an uniform rate of profit and unit wage, are the same that guarantee that *each firm* in such an industry is able to do so, whatever the level of scale it is operating. In fact, considering the general notation of previous chapters, if a_{11}, a_{21}, a_{31} and l_1 are the amount of commodities and labor necessary for a firm in industry 1 to produce one unit of commodity , the constant returns to scale imply that the amount of commodities and labor necessary to produce α^i units in commodity will be $\alpha^i a_{11}, \alpha^i a_{21}, \alpha^i a_{31}$ and $\alpha^i l_1$, and the total amount used in the industry will be $Sa_{11}, Sa_{21}, Sa_{31}$ and Sl_1, where S it the total amount produced in the industry, i.e. $\sum_{i=1}^{\texttt{N\_firms}} \alpha^i$. This means that if p_1, p_2, p_3 and w attain balanced budget in a firm of industry 1 producing one commodity (the rate of profit, to avoid confusion with prices, is denoted r)

$$(p_1 a_{11} + p_2 a_{21} + p_3 a_{31})(1 + r) + l_1 w = p_1 \tag{8.4}$$

so they do when the firm is producing at level α^i

$$\alpha^i(p_1a_{11} + p_2a_{21} + p_3a_{31})(1 + r) + \alpha^i l_1 w = \alpha^i p_1 \tag{8.5}$$

and so they do for the industry in its whole

$$\sum_{i=1}^{N} \alpha^i(p_1a_{11} + p_2a_{21} + p_3a_{31})(1 + r) + \sum_{i=1}^{N} \alpha^i l_1 w = \sum_{i=1}^{N} \alpha^i p_1 \tag{8.6}$$

$$\Rightarrow S(p_1a_{11} + p_2a_{21} + p_3a_{31})(1 + r) + Sl_1 w = Sp_1 \tag{8.7}$$

Notice that from expression 8.6 it is also clear that a_{11}, a_{21} and a_{31} are the production coefficients of the economy (whatever the scale of production).

From this considerations two important results follow: first of all, the prices and wage determined by Sraffa's equation will be the same, regardless the increase or decrease in the scale of production of the various industries; moreover, no firm will ever be in trouble in finding the necessary resources to acquire the means of production necessary to operate at its current scale. Indeed, Sraffa's prices prices are *just those* prices that guarantee that this is possible.
One final consideration regarding the assumption of constant return to scale: increasing the scale of production by a factor of one will also result in an increase in the profits earned by the firm, so that the assumption that firms invest as much as they can is consistent with a profitability criterion in firm's choices.

One of the main advantage of the introduction of prices is that we are finally able to aggregate heterogeneous commodities. Therefore we can compute the usual macro-aggregates of interest and store them into newly created global variables: `Value_production`, `Net_product`, `Investment`, `Consumption`, `Wage_share` and `Profit_share`.

8.3 Procedures

Let us now consider how we have modified the procedures with respect of our previous models.

The `setup` procedure does not change particularly with respect of previous versions of the model. It simply includes the procedure `set_max_rate`, `set_profit` and `set_price` described in the previous section. As we have said, indeed, those variable are not, with the assumption of constant return to scale, influenced by the scale of production.

The `produce` procedure is exactly identical to the one described before.

The `realization` procedure, on the other hand, is introduced in this version. The firms are assigned a new variable, `value`, that, for each firm, takes the value of the size

of the product between the size of its production (the scale at which is operating) and the price of the kind of commodity it is producing. Since we know that the firms will be able to find a buyer for what they are selling, it is as they were already assigned the nominal value of their production. Moreover, in this procedure we compute the variable `Value_production`. The procedures is therefore

```
to realization
  ask firms [if sector = "iron"  [set value s * p_i]
             if sector = "coal"  [set value s * p_c]
             if sector = "wheat" [set value s * p_g]]
set Value_production sum [value] of firms
end
```

`find_inputs_2` is a procedure that is very similar to the `find_inputs` procedure of previous models. Indeed, since we know that, given those prices, all firms will be able to replace their means of production, they just simply ask other firms to give them what they need, as in the first model. The only difference is that they subtract from their `value` the value of the commodity they receive.

Also the `surplus` procedure is identical to before, with the addition of the computation of the variable `Net_product`

In the `pay_wages` procedure the firms simply subtract from their `value` the amount of wages they have to give to the workers they hired. The amount of `value` left to the firm is therefore its profit, so we can compute the aggregate variables `Wage_share` and `Profit_share`, consisting in the total share of the value of the Net Product going respectively to workers and firms.

Finally, the `invest_value` is the extension of the previously used `invest` procedure where the firms are asked to check also if they have enough value left to "pay" for increased means of production. Therefore the firms first checks if there are enough physical commodities to sustain the production of a further unit. If this is not the case, as before, it asks to all firms in its industry to stop searching (of course, an other implication of constant return to scale). If there are enough resources, then the firms cosider if their realized profit is enough for paying for such resources, meaning check if the following condition is respected:

```
if-else  ((iron_need /s)* p_i +(coal_need /s) * p_c +(wheat_need /s)* p_g) < value
```

If this is the case, they appropriate the resources and adjourn their variables as in the procedure `invest`, and they subtract the value of the acquired resources from their `value`. In this procedure we determine also the value of the Net Product that has been employed as investment into the global variable `Investment` and the remaining part, comprehending all wages and that part of profits that has not been invested, into the global variable `Consumption`

Let us now consider the evidence resulting by the observation of an economy working according to the procedures listed above, which are gathered in the procedure `go_prices`

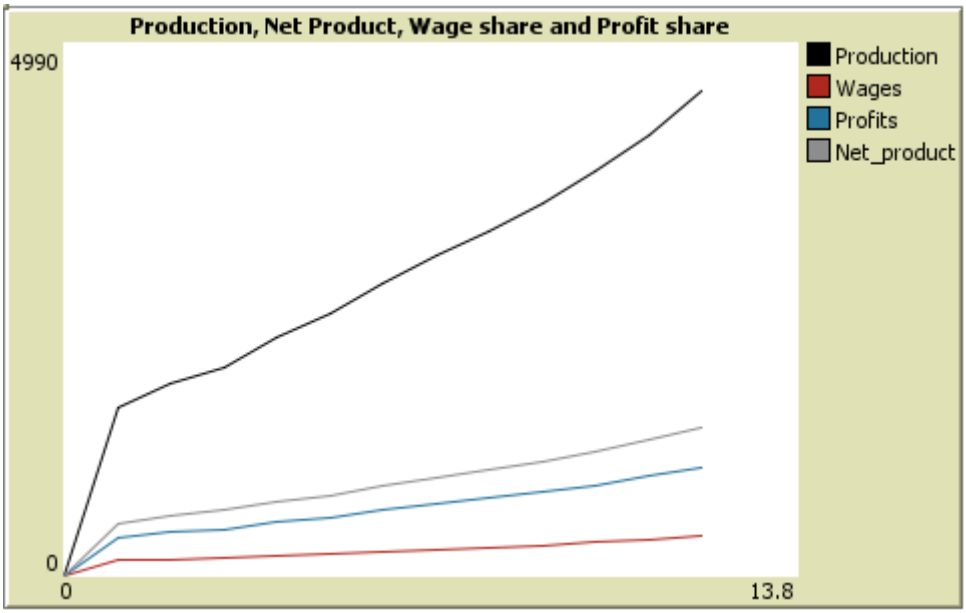
```
to go_prices
  produce
  realization
  find_inputs_2
  surplus
  pay_wages_value
  check_distribution
  invest_value
  uses
  tick
end
```

8.4 Evidence

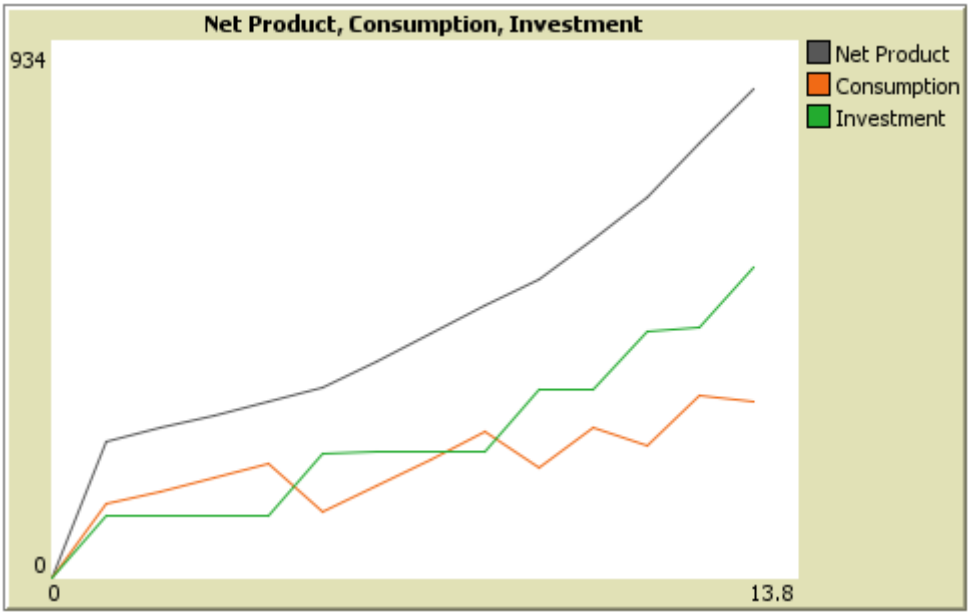
From a series of graphs in the Interface we can keep track of the evolution of the main macro-economic aggregate in an economy which starts in a self-replacing state, as well as experimenting how those aggregates are influenced by the matrix of technical coefficients, by the determination of the exogenous distributive variable and by chance.

`Value_Production` and `Net_Product`. Since the prices remain constant for every possible level of scale, every increase in the value of production stands for an increase in the amount of commodities produced. This will grow, of course, depending on the amount of investment that is done each year.

`Wage_share` and `Profit_share` do not present much surprises. The repartition of the Net Product is determined outside of the production process, and therefore the aggregate distribution variables will follow the path of the Net Product. Of course a decrease / increase in the exogenous profit rate `p` would, keeping identical the production coefficients, result in a decrease / increase in the `Profit_share` at benefits of `Wage_share`.
In the following graph we can see an example of the evolution of the above variables in a growing economy.



The evolution of the variables `Consumption` and `Investment` is the most interesting to analyse, since from the dynamic of investment depends the growing of the economy. The subdivision of the Net Product between investment and consumption is apparently much less predictable than the one between wage share and profit share, and can change substantially during the growth path, as we can see from the graph below.



Since, as we have seen, the part of consumption that derives from workers expenditures is rather stable over time, an increase in consumption can only mean that firms are

not able to invest a greater part of their profits. This, as mentioned before, can happen for two reasons: the economy does not produce enough surplus of all commodities to sustain an increase in production, i.e. the surplus generated for one commodity is not very high so that few firms exhaust it rapidly, leaving the others in the impossibility to invest; the value left to the firm after having recovered the means of production for its current scale and paid the wage is not enough to buy further commodities.

With regard to the first case, it is a purely physical constraint of the economy and we can recall the considerations made for the purely physical model of the previous section. But even if the economy is capable of producing plenty of commodities, there could be "financing" problems. In our model, indeed, there is no possibility of getting credit from other agents or from outside the production system. The only resource available for a firm is its profit. Therefore, once that of this profit there is left less than the amount necessary to buy the means of production to produce one unit more, the firm cannot invest any more.

The amount of resources that a firm will spend in investment will be strictly related to the profits realized, and therefore on the uniform rate of profit $\mathtt{p}$. Actually it easily shown that the condition whether a firm can invest in *just one* further unit of production depends uniquely on the profit rate $\mathtt{p}$ and the scale at which the firm is operating. Indeed, using standard notation, the profit realized by a firm producing commodity 1 at scale α is

$$\alpha r(p_1 a_{11} + p_2 a_{21} + p_3 a_{31}) \tag{8.8}$$

while the value of the commodities necessary to produce one unit more of commodity 1 is

$$(p_1 a_{11} + p_2 a_{21} + p_3 a_{31}) \tag{8.9}$$

This means that a firm will be able to invest if

$$\alpha r(p_1 a_{11} + p_2 a_{21} + p_3 a_{31}) \geq (p_1 a_{11} + p_2 a_{21} + p_3 a_{31}) \tag{8.10}$$

or

$$r \geq \frac{1}{\alpha} \tag{8.11}$$

The above expression indicates that if the rate of profit or the operating scale are too low, the firm will not have the resource to increase its production at all.
This consideration led us to reconsider the initial scale of production of firms, since keeping it at 1 would have required a rate of profit too high (equal to 1, indeed) for most actual economies.

8.5 Periodical Bargaining

As in the purely physical model, we analyse also in the case of determination of prices the effect of a periodical bargaining over the distributive variable.
In order to do so we consider that at the beginning of our analysis the distribution is

completely in favour of firms, which receive as profit the whole Net Product. Then, as long as the unemployment goes down, the bargaining power of the workers raises, and so they are able to lower the profit rate `p`.[4] The new prices are then computed, and the process starts over again.
In order to do so we add line of code in the `setup` procedure, setting the new variable `emp_t0` as equal to the actual level of unemployment.

```
set emp_t0 count workers with [job = 1] / count workers
```

We create then the procedure `bargain`, which can be activated in the Interface by an apposite switch

In such procedure, we determine the value of a new global variable, `delta_emp`, which represent the increase in the employment rate with respect to the initial situation. We then set the variable `profit` (that, we remember, represent the fraction of the maximum rate of profit `R` that constitutes the actual rate of profit `p`) as one minus `delta_emp`. We then compute the new values of `p`, the prices and wage.

```
to bargain

  set delta_emp count workers with [job = 1] / count workers - emp_t0
  set profit 1 - delta_emp
  set_profit
  set_prices

end
```

Notice that the variable `delta_emp` assumes values between 0 (at the initial situation, when the employment rate is equal to `emp_t0`) and `1-emp_t0` when there are no unemployed left. Therefore the variable `profit` can assume values between 1 (at the initial situation) and `1-(1-emp_t0)= emp_t0` when we have full employment, decreasing continuously for all the intermediate values of employment.

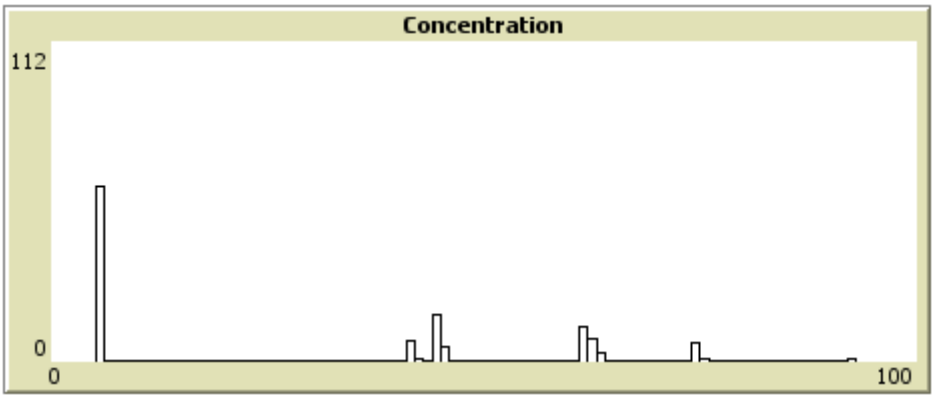
Let us consider the effects of this on the other variables in an economy which starts in self-replacing state. From our discussion above the effects should be clear .

We know for sure that the `wage` will increase, following a decrease of the rate of profit, as discussed in Chapter 4, section 4.3.3. The effect on prices, on the other hand, are not predictable, and depends on the relative intensities of capital and on the commodity

[4]Notice that it would make more sense economically to set the wage as the exogenous variable, and computing endogenously the rate of profit. Unfortunately Netlogo does not offer the possibility of solving systems of non-linear equation, so we will stick with the procedure, analytically equivalent, that we have considered until now of setting the rate of profit as exogenous.

chosen as *numeraire*, as discussed regarding procedure `evolution_prices` .

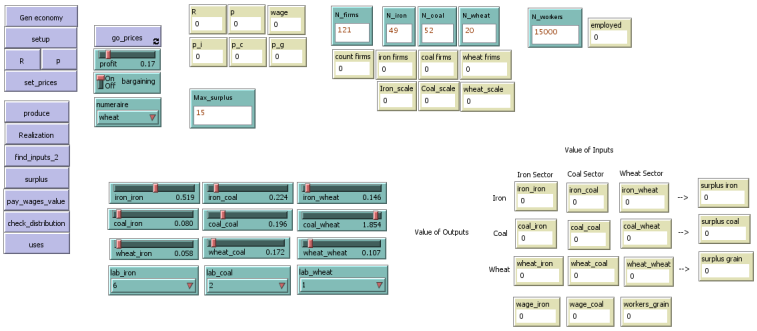
It is worth to spend some words on the effects that a progressive increase in the rate of profit would have on aggregate investment. Since and increase in `Wage_share` clearly reduces the `Profit_share`, it will surely have the effect of reducing the resources available for investment. Anyway, for certain initial level of the rate of profit, we may have also an other interesting result. We have discussed in the previous section how condition 8.11 must be satisfied for a firm to be able to invest. But since the rate of profit decreases progressively, and the scale at which each firm operates depends eventually on chances, it is possible that the rate of profit lowers to a level such that some firms are prevented from investing again (those with a lower scale) while some others still have enough resources (those with a greater scale). The "concentration" of firms as discussed with respect to the first, physical model will appear therefore with some "holes", as in the image below.



This can be seen as representative of the difficulties that, in word where it is difficult to obtain loans (actually, impossible), smaller enterprises must face in order to finance their activities, while greater corporation can overcome these difficulties thanks to their relative abundance of liquidity.

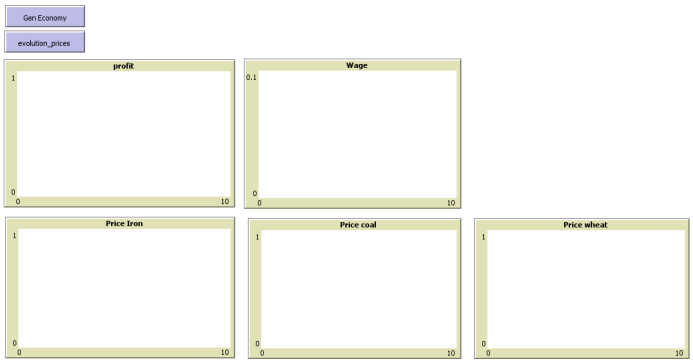
8.6 Interface

The complete interface of the model results therefore as following.



Graphs

Changes in variables



Chapter 9

Conclusions

In the dissertation we have analysed, both through the works of influential thinkers and through the building of Agent-Based Models, the vision of the production process as a circular flow, in which the outcome does not depend on the initial endowments of factors of production, but from the complex inter-industrial relationship between industries in an economy in which commodities are produced by means of other commodities. This approach allowed us to reason over the technical and social necessities of a system that needs to reproduce itself, rather than on the outcome resulting from the individual actions of self-motivated agents.

In the first part of the dissertation we have reviewed the works of three major scholars of the 20th century which share this conception of the production process.

From Leontief's input-output approach we have analysed the mathematical and technical conditions requested for an economy to be able of reproducing itself and generate a positive surplus in each sector. Moreover we have considered an expansion of his model which allows for a constant rate of growth.

The work of Piero Sraffa expands this approach analysing in greater depth the process of distribution of the surplus potentially generated among different social classes. Therefore, Sraffa's approach considers also social requirements (as the payments of an uniform wage and rate of profit) in addition to the technical necessities of an economic system.

Finally, Von Neumann's model analysed the characterizations of a quasi-dynamic equilibrium of an economy with a circular production process and very simplifying assumption over the distribution of income.

The three thinkers share other characteristics beside the obvious one of the analysis of the production process as a circular flow. All three do not investigate the role of the demand in the formation of the exchange values. Actually they, more or less explicitly, disregard the role of psychology and individual choice in the determination of the "equilibrium" prices or quantities. This approach makes sense if we consider that all three authors dealt with what Pasinetti (2005) calls "pure economic analysis". Pasinetti distinguish between two stages of economic theory: a "pure" level, intended as an abstract

exercise aimed at analysing those characteristics of an economic system which remain rather invariant over time; and a "full" economic analysis, which has to include also those aspects which are much more inclined to vary, as human behaviour. The consideration of an economic system as a circular flow is surely included in the first category, as it is the analysis of the physical and social necessities that an economic system needs to satisfy in order to reproduce itself. In theory, the results derived by this kind of analysis should be able to be consistent with a broad range of individual behaviour and institutional setting.

In the second part of the dissertation we try to reason over the theoretical framework analysed in the first part with the help of three different Agent-Based Models.

In the first model, which consider a very simple economy where the product of the single firms is donated to the community, we reason over the requirements that the technologies of production and the proportions among the outputs of the various industries need to satisfy in order to permit the sustainability of the system. We consider also the influence of the choices of the agents, that we consider to be determined by chance, and how they may influence the growing path of an economy.

In the second model we consider the presence of a different kind of agent, workers, whose labor is necessary for the firms to produce. Given that the quantity of labor is, in general, not reproducible as the quantity of regular commodities, the presence of a scarce resource set some limits to the possible expansion of an economy. Moreover, we have analysed how the presence of hypothesis regarding the spending of the "income" would affect the growing path. Moreover we have also analysed the possible imbalances derived from and adversarial repartition of the surplus in presence of incomplete information.

In the third model we have analysed the introduction of Sraffa's prices. The possibility of aggregating heterogeneous measure has permitted a simpler analysis of the macroeconomic variables, even with the vagueness related to the possible choices of different commodities as *numeraire*, as pointed out by Sraffa. We also reason over the relationship between the determination of the distributive variables and the possibility of a firm to afford an investment, which depends on the scale at which the firm is operating.

9.1 Hints for future steps

There are many possibilities for further line of research, taking advantage of the power that Agent Based Models have as a reasoning and teaching tool.

One possibility could be to compare Sraffian "necessary" prices with prices arising from a different, bottom-up approach, as for example the double auction mechanism described by Straatman *et al.* (2013). The main difficulty on this investigation would be to determine a decentralized mechanism that would be coherent with the assumption of uniform rate of profit, on whose economic meaning there have been disagreement (see Boggio (1990) but also Sinha (2014)).

An other, interesting possibility could be to analyse the evolution of non-constant return to scale. As we have mentioned many times, Sraffa does not make this assumption, since he is only concerned with those properties of economic systems that do not depend

on a variation in quantity. Considering, as we do, growing economies would require therefore a periodical redefinition of what are the necessary prices for the reproduction of the system, which would have an effect on the accumulation path. The difficulty related to this line of research is the complexity of a logically sound theory regarding non-constant return, in particular in a setting like ours without naturally scarce resources.

Part III

Mathematical Appendix

Chapter 10

Perron–Frobenius Theorems

10.1 Introduction

A matrix $\mathbf{A}$ is said to be *positive* if

$$\forall i, j \quad a_{ij} > 0 \tag{10.1}$$

where a_{ij} is an element of $\mathbf{A}$. We write then $\mathbf{A} > 0$.
A matrix $\mathbf{A}$ is said to be *non-negative* if

$$\forall i, j \quad a_{ij} \geq 0 \tag{10.2}$$

If it is possible that $\mathbf{A} = \mathbf{0}$ we write $\mathbf{A} \geqq \mathbf{0}$. If at least one element of $\mathbf{A}$ must be positive we write $\mathbf{A} \geq \mathbf{0}$.
A square matrix $\mathbf{A}$ is said to be *reducible* if, via a permutation, it is possible to reduce it to the following form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{0} & \mathbf{A_{22}} \end{bmatrix} \tag{10.3}$$

Where A_{11} and A_{22} are square matrices and $\mathbf{O}$ is the null matrix. If this is not possible, then the matrix is said to be *irreducible*.
Note that A_{11} and A_{22} can be reducible as well, so that a reducible matrix can be always reduced to the block quasi-triangular form.

$$\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} & \dots & \mathbf{A_{1n}} \\ \mathbf{0} & \mathbf{A_{22}} & \dots & \mathbf{A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{A_{nn}} \end{bmatrix} \tag{10.4}$$

Where A_{11},..., A_{22} are irreducible matrices, not necessarily of the same order.

Let us now consider a series of propositions regarding eigenvalues and eigenvectors of non-negative matrices, mainly due to *Perron* and *Frobenius* and generally are known as

Perron-Frobenius proposition(s). These proposition have a stronger version, valid only for irreducible matrices, and a weaker version, which applies to all non-negatives matrices. Following Pasinetti (1975) we will provide demonstration of only the stronger version, and we will merely state teh weaker version.

10.2 Irreducible Matrices

Lemma 10.2.1. *Let $\boldsymbol{A}$ be a (n, n) non-negative, irreducible matrix. Then*

$$(\mathbf{I} + \mathbf{A})^{\mathbf{n}} > \mathbf{0} \tag{10.5}$$

Proof. Let $\mathbf{x} \geq \mathbf{0}$.[1]It is always possible to re-write vector $\mathbf{x}$, via permutation of the indexes of its elements, in the following form

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} \tag{10.6}$$

where $\mathbf{y}$ is a vector of dimension r with only positive components.
Applying the *same* permutation used to transform $\mathbf{x}$ in $\mathbf{y}$ to the matrix $\mathbf{A}$, we can decompose it in the form

$$\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix} \tag{10.7}$$

whhere $\mathbf{A_{11}}$ and $\mathbf{A_{22}}$ are square matrices of order r and $n - r$ respectively. We can then rewrite $(\mathbf{I} + \mathbf{A})\mathbf{x} = \mathbf{x} + \mathbf{A}\mathbf{x}$ as

$$\begin{aligned} \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} &= \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{A_{11}y} + \mathbf{0} \\ \mathbf{A_{21}y} + \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{y} + \mathbf{A_{11}y} \\ \mathbf{A_{21}y} \end{bmatrix} \end{aligned}$$

Now, the first r components of the above vector are strictly positive since $\mathbf{y}$ is positive. At the same time, we know that it cannot be the case that

$$\mathbf{A_{21}y} = \mathbf{0} \tag{10.8}$$

since it would be a contradiction with the hypothesis of irreducibility of $\mathbf{A}$.
This means that $(\mathbf{I} + \mathbf{A})\mathbf{x}$ has *at least* $r + 1$ positive components, at least one more than x. Repeating the process using $(\mathbf{I} + \mathbf{A})\mathbf{x}$ instead of x allow us to show that $(\mathbf{I} + \mathbf{A})^{\mathbf{2}}\mathbf{x}$ has at least $r + 2$ positive components. Therefore repeating the process n times show us that

$$(\mathbf{I} + \mathbf{A})^{\mathbf{n}}\mathbf{x} > \mathbf{0} \tag{10.9}$$

And, since x can be chosen arbitrarily among non-negative vectors the Lemma is proved □

[1]Remember that this means that at least an element of $\mathbf{x}$ must be positive.

Before the next Lemma, let us make some considerations:

Consideration I: $\forall \lambda < \lambda(\mathbf{x}) \Rightarrow \mathbf{Ax} > \lambda\mathbf{x}$.

Consideration II: $(\mathbf{I}+\mathbf{A})^{\mathbf{n}}\mathbf{A} = \mathbf{A}(\mathbf{I}+\mathbf{A})^{\mathbf{n}}$.

Consideration II: given Lemma 10.2.1 we have that $\forall \mathbf{A} \geq \mathbf{0}, \mathbf{x} \geq \mathbf{0}$ the following hold

$$(\mathbf{I}+\mathbf{A})^{\mathbf{n}}\mathbf{Ax} > \mathbf{0}, \quad (\mathbf{I}+\mathbf{A})^{\mathbf{n}}\mathbf{x} > \mathbf{0}$$

Define then the function $\lambda(\mathbf{x}) = \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ such that

$$\lambda(\mathbf{x}) := \max\{\lambda \in \mathbb{R} : \mathbf{Ax} \leq \lambda\mathbf{x}\} \tag{10.10}$$

Note that this means that

$$(Ax)_i \geq \lambda x_i \quad i = 1, .., n \tag{10.11}$$

And that therefore

$$\lambda(\mathbf{x}) = \min_{i, x_i > 0} \frac{(Ax)_i}{x_i} \tag{10.12}$$

We can now proceed with the following Lemma

Lemma 10.2.2. *The function $\lambda(\mathbf{x})$ admits a positive maximum in $\mathbb{R}^n_{\geq 0}$.*

Proof. Let V be the subset of $\mathbb{R}^n_{\geq 0}$ such that

$$\mathbf{x} \in V \quad \text{if} \quad \sum_{i=1}^{n} x_i = 1 \tag{10.13}$$

Consider the function $F(\mathbf{x} = (I+A)^n\mathbf{x}$. Now consider the image of V under F. the set $F(V)$ is compact and the function $\lambda(.)$ is continuous over $F(V)$. Therefore because of Weierstrass proposition the function $\lambda(.)$ achieves a maximum value in $F(V)$. Let us denote it with λ_{max}
Now consider that, given Lemma 10.2.1 , $\forall \mathbf{x}, \mathbf{y} \geq \mathbf{0}$, if $\mathbf{x} \geq \mathbf{y} \Rightarrow (\mathbf{I}+\mathbf{A})^{\mathbf{n}}\mathbf{x} \geq (\mathbf{I}+\mathbf{A})^{\mathbf{n}}\mathbf{y}$. So, by definition of function $\lambda(.)$

$$\lambda(\mathbf{x})\mathbf{x} \leq \mathbf{Ax} \tag{10.14}$$

$$(\mathbf{I}+\mathbf{A})^{\mathbf{n}}\lambda(\mathbf{x})\mathbf{x} \leq (\mathbf{I}+\mathbf{A})^{\mathbf{n}}\mathbf{Ax} \tag{10.15}$$

$$\lambda(\mathbf{x})(\mathbf{I}+\mathbf{A})^{\mathbf{n}}\mathbf{x} \leq \mathbf{A}(\mathbf{I}+\mathbf{A})^{\mathbf{n}}\mathbf{x} \qquad \text{(Consideration II)} \tag{10.16}$$

$$\lambda(\mathbf{x})[(I+A)^n x]_i \leq [A(I+A)^n x]_i \qquad for \quad i = 1, .., n \tag{10.17}$$

$$\lambda(\mathbf{x}) \leq \frac{[A(I+A)^n x]_i}{[(I+A)^n x]_i} \qquad for \quad i = 1, .., n \tag{10.18}$$

But notice that

$$\lambda((\mathbf{I}+\mathbf{A})^{\mathbf{n}}) = \min_{i,x_i>0} \frac{[A(I+A)^n x]_i}{[(I+A)^n x]_i} \tag{10.19}$$

And therefore $\lambda((\mathbf{I}+\mathbf{A})^{\mathbf{n}}) \geq \lambda(\mathbf{x}) \quad \forall \mathbf{x}$. Therefore λ_{max} is the maximum in all $\mathbb{R}^n_{\geq 0}$ □

Proposition 10.2.3. *Let λ_{max} be the positive max of function $\lambda(\mathbf{x})$. Let $\boldsymbol{A}$ be a (n,n) non-negative, irreducible matrix. Then*

i) λ_{max} is an eigenvalue of $\boldsymbol{A}$, or

$$\mathbf{Ap} = \lambda_{max}\mathbf{p} \tag{10.20}$$

ii) To λ_{max} are associated positive right-eigenvector $\boldsymbol{p}$ and left-eigenvector $\dot{\mathbf{p}}$, or

$$\mathbf{p} > \mathbf{0}, \quad \dot{\mathbf{p}} > \mathbf{0} \tag{10.21}$$

Proof. Let us demonstrate the two parts separately.

i) In the first place, let us notice that the following expression cannot hold for any $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$

$$\lambda_{max}\mathbf{x} < \mathbf{Ax} \tag{10.22}$$

In fact since λ_{max} is the maximum value of $\lambda(\mathbf{x})$, it holds $\lambda_{max} \geq \lambda(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^n_{\geq 0}$ and so also

$$\lambda_{max}\mathbf{x} \geq \lambda(\mathbf{x})\mathbf{x} \tag{10.23}$$

But 10.22 and 10.23 together would mean

$$\mathbf{Ax} > \lambda(\mathbf{x})\mathbf{x} \tag{10.24}$$

and this would be in contradiction with the definition of $\lambda(.)$, since it would be possible to find a $\lambda(\mathbf{x}) + \epsilon$ such that the condition of 10.10 is satisfied.

Now, let $\tilde{\mathbf{x}}$ be the argmax of function $\lambda(.)$. Then by definition we have

$$\lambda_{max}\tilde{\mathbf{x}} \leq \mathbf{A}\tilde{\mathbf{x}} \tag{10.25}$$

$$(\mathbf{I}+\mathbf{A})^{\mathbf{n}}\lambda_{max}\tilde{\mathbf{x}} \leq (\mathbf{I}+\mathbf{A})^{\mathbf{n}}\mathbf{A}\tilde{\mathbf{x}} \tag{10.26}$$

$$\lambda_{max}(\mathbf{I}+\mathbf{A})^{\mathbf{n}}\tilde{\mathbf{x}} \leq \mathbf{A}(\mathbf{I}+\mathbf{A})^{\mathbf{n}}\tilde{\mathbf{x}} \tag{10.27}$$

Let us call $\mathbf{p} = (\mathbf{I}+\mathbf{A})^{\mathbf{n}}\tilde{\mathbf{x}}$. We can then rewrite expression 10.27 as

$$\lambda_{max}\mathbf{p} \leq \mathbf{Ap} \tag{10.28}$$

Now, we know from the discussion relative to expression 10.22, since $\mathbf{p} \neq \tilde{\mathbf{x}}$ theat we cannot have

$$\lambda_{max}\mathbf{p} < \mathbf{Ap} \tag{10.29}$$

We are left with the possibility that the two expression are equal and that they differ for just some element. This correspond on determining if for the vector $\mathbf{z} = \mathbf{Ap} - \lambda_{max}\mathbf{p}$ holds

$$\mathbf{z} \geq \mathbf{0} \quad \text{or} \quad \mathbf{z} = \mathbf{0} \tag{10.30}$$

By the proof of Lemma 10.2.1 we know that if $\mathbf{z} \geq \mathbf{0}$ then right-multiplying for $(\mathbf{I} + \mathbf{A}^n$ we have

$$(\mathbf{I} + \mathbf{A})^{\mathbf{n}}\mathbf{z} > \mathbf{0} \tag{10.31}$$

$$(\mathbf{I} + \mathbf{A})^{\mathbf{n}}(\mathbf{Ap} - \lambda_{max}\mathbf{p}) > \mathbf{0} \tag{10.32}$$

$$(\mathbf{I} + \mathbf{A})^{\mathbf{n}}\lambda_{max}\mathbf{p} < (\mathbf{I} + \mathbf{A})^{\mathbf{n}}\mathbf{Ap} \tag{10.33}$$

$$\lambda_{max}(\mathbf{I} + \mathbf{A})^{\mathbf{n}}\mathbf{p} < \mathbf{A}(\mathbf{I} + \mathbf{A})^{\mathbf{n}}\mathbf{p} \tag{10.34}$$

$$\tag{10.35}$$

But this would mean that vector $\mathbf{A}(\mathbf{I} + \mathbf{A})^{\mathbf{n}}\mathbf{p}$ would respect expression 10.22, and since $\mathbf{A}(\mathbf{I} + \mathbf{A})^{\mathbf{n}}\mathbf{p} \neq \tilde{\mathbf{x}}$ this is a contradiction. We have therefore that $\mathbf{z} = \mathbf{0}$, or that

$$\mathbf{Ap} = \lambda_{max}\mathbf{p} \tag{10.36}$$

which means that p i the right-eigenvector associated with λ_{max}

ii) To show that $\mathbf{p} > \mathbf{0}$ let us assume by contradiction that $p_i = 0$ for some i. But then it would be possible, as in the demonstration of Lemma 10.2.1 , to rewrite expression 10.36 as

$$\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} = \lambda_{max} \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} \tag{10.37}$$

which means

$$\begin{cases} \mathbf{A_{11}y} = \lambda_{max}\mathbf{y} \\ \mathbf{A_{21}y} = \mathbf{0} \end{cases} \tag{10.38}$$

from which it would result that $\mathbf{A_{21}} = \mathbf{0}$, since by construction $\mathbf{y} > \mathbf{0}$, and this would be in contradiction with matrix $\mathbf{A}$ being irreducible.

Note that the same demonstration could have been done for the left-eigenvector $\dot{\mathbf{p}}$

□

Consideration IV. Let $\mathbf{B}$ be a non-negative square matrix of order n, λ one of his eigenvalues and $\mathbf{x}$ the corresponding left-eigenvalue. By definition

$$\mathbf{p}'\mathbf{B} = \lambda\mathbf{x}' \tag{10.39}$$

which can be rewritten as

$$\sum_{i=1}^{n} x_i b_{ij} = \lambda x_j \qquad j = 1, .., n \tag{10.40}$$

By the triangle inequality we have that

$$\sum_{i=1}^{n} \mid x_i \mid b_{ij} \geq \mid \lambda \mid\mid x_j \mid \qquad j = 1, .., n \tag{10.41}$$

Defining $\mathbf{x}* = (\mid x_1 \mid, ..., \mid x_n \mid)$ we can rewrite he above equation as

$$\mathbf{x} * \mathbf{B} \geq \mid \lambda \mid \mathbf{x}* \tag{10.42}$$

Proposition 10.2.4. *Let $\boldsymbol{A}$ be a (n, n) non-negative, irreducible matrix. Let λ be an eigenvalue of $\boldsymbol{A}$. Then*

$$\lambda_{max} \geq \mid \lambda \mid \tag{10.43}$$

Proof. Consider a matrix $\mathbf{B}$ such that

$$\mathbf{0} \leq \mathbf{B} \leqq \mathbf{A} \tag{10.44}$$

Let λ be an eigenvalue of $\mathbf{B}$ and $\mathbf{x}$ the correspondent left-eigenvector.
By expressions 10.42 and 10.44 we have that

$$\mathbf{x} * \mathbf{A} \geqq \mathbf{x} * \mathbf{B} \geqq \mid \lambda \mid \mathbf{x}* \tag{10.45}$$

Right-multiplying the 10.45 by $\mathbf{p}$ and using the definition of λ_{max}

$$\mathbf{x} * \mathbf{A}\mathbf{p} \geq \mid \lambda \mid \mathbf{x} * \mathbf{p} \tag{10.46}$$

$$\lambda_{max}\mathbf{x} * \mathbf{p} \geq \mid \lambda \mid \mathbf{x} * \mathbf{p} \tag{10.47}$$

Since by Proposition 10.2.3 $\mathbf{p} < 0$, it follows

$$\lambda_{max} \geq \mid \lambda \mid \tag{10.48}$$

In particular, setting $\mathbf{B} = \mathbf{A}$ the proposition is proved. □

Lemma 10.2.5. *Let $\boldsymbol{A}$ be a (n, n) non-negative, irreducible matrix and $\boldsymbol{B}$ a non-negative matrix such that $\mathbf{0} \leq \mathbf{B} \leqq \mathbf{A}$. Let λ be an eigenvalue of $\boldsymbol{B}$. Then*

$$\mid \lambda \mid = \lambda_{max} \Rightarrow \mathbf{B} = \mathbf{A} \tag{10.49}$$

Proof. Given $\mid \lambda \mid = \lambda_{max}$ expression 10.45 become

$$\mathbf{x} * \mathbf{A} \geqq \mathbf{x} * \mathbf{B} \geqq \lambda_{max}\mathbf{x}* \tag{10.50}$$

Consider now the possibility that

$$\mathbf{x} * \mathbf{A} > \lambda_{max}\mathbf{x}* \tag{10.51}$$

which means that the two vectors differ for some elements. If this were the case, then we could, as in the case in the proof of Proposition 10.2.6, find a vector such that expression

10.22 would hold (right-multiplying for $\mathbf{I} + \mathbf{A}^n$ and rearranging, as in 10.31-10.35.
But this means that the two extreme of expression 10.50 are equal to each other, and so also to the central member.
From

$$\mathbf{x} * \mathbf{A} = \lambda_{max}\mathbf{x}* \tag{10.52}$$

we can see that $\mathbf{x}*$ is the right-eigenvector associated with λ_{max}, and therefore $\mathbf{x}* > \mathbf{0}$.
Then from

$$\mathbf{x} * \mathbf{A} = \mathbf{x} * \mathbf{B} \tag{10.53}$$

$$\mathbf{x} * (\mathbf{A} - \mathbf{B}) = \mathbf{0} \tag{10.54}$$

it follows that $\mathbf{A} = \mathbf{B}$ □

Proposition 10.2.6. *Let $\boldsymbol{A}$ be a (n, n) non-negative, irreducible matrix and λ^A_{max} its maximum eigenvalue. Then λ_{max} is a continuous, increasing function of the coefficients of $\boldsymbol{A}$*

Proof. Consider a square non-negative matrix $\mathbf{B}$ such that $\mathbf{0} \leq \mathbf{B} \leq \mathbf{A}$. Notice that this time we have assumed that $\mathbf{B} \neq \mathbf{A}$. Then by the proof of Proposition 10.2.7 we have that

$$\lambda^A_{max} \geq \lambda^B_{max} \tag{10.55}$$

In particular, because of Lemma 10.2.5, we cannot have that $\lambda^A_{max} = \lambda^B_{max}$, or we would have $\mathbf{B} = \mathbf{A}$, a contradiction. It must be the case then that

$$\lambda^A_{max} > \lambda^B_{max} \tag{10.56}$$

Therefore λ_{max} is an increasing function of the coefficients of $\mathbf{A}$.

The continuity is assured by the continuity of eigenvalues wrt the coefficients of the correspondent matrix (the function from the matrix to its characteristic polynomial is continuous, and so it is the function from a polynomials to its roots, the eigenvalues). □

From this proposition we have two interesting corollaries

Corollary 10.2.6.1. *Let $\boldsymbol{A}$ be a (n, n) non-negative, irreducible matrix and λ^A_{max} its maximum eigenvalue. Let $\mathbf{A_{ij}}$ be a square sub-matrix of $\boldsymbol{A}$. Then*

$$\lambda^A_{max} > \lambda^{A_{ij}}_{max} \tag{10.57}$$

Proof. Let us add rows and columns of zeros to $\mathbf{A_{ij}}$ in order to make it the same dimension of $\mathbf{A}$. Then $\mathbf{A_{ij}}$ become a matrix $\mathbf{B}$ analogous of that one in the proof of Proposition 10.2.6, from which the result. □

Corollary 10.2.6.2. *Let $\boldsymbol{A}$ be a (n, n) non-negative, irreducible matrix and λ^A_{max} its maximum eigenvalue. Then λ^A_{max} is a simple root of the characteristic polynomial of $\boldsymbol{A}$*

$$\mathbb{P}(A, \lambda) = Det[\lambda I - A] = \mathbf{0} \tag{10.58}$$

Proof. Let $\mathbf{A_i}$ be the sub-matrix of $\mathbf{A}$ obtained by deleting the *i-th* row and column of $\mathbf{A}$. The derivative of the function $\mathbb{P}(A, \lambda) = Det[\lambda I - A]$ is equal to

$$\frac{\delta \mathbb{P}(A, \lambda)}{\delta \lambda} = \sum_{i=1}^{n} \mathbb{P}(A_i, \lambda) \tag{10.59}$$

Any of the n $\mathbb{P}(A_i, \lambda)$ can be made equal to zero substituting to λ one of the eigenvalues of the correspondent $\mathbf{A_{ij}}$, but by Corollary 10.2.6.1 λ^A_{max} is different (strictly greater) that all of them. Therefore λ^A_{max} is not a root of the first derivative of $\mathbb{P}(A_i, \lambda)$, that is necessary and sufficient condition for λ^A_{max} to be simple root of the characteristic polynomial □

Proposition 10.2.7. *Let $\boldsymbol{A}$ be a (n, n) non-negative, irreducible matrix and λ_{max} its maximum eigenvalue. To any other eigenvalue λ of $\boldsymbol{A}$ different from λ_{max} is associated an eigenvector $\mathbf{x}^\lambda \neq \mathbf{0}$ with at least one negative component.*

Proof. By definition we have

$$\mathbf{A}\mathbf{x}^\lambda = \lambda \mathbf{x}^\lambda \tag{10.60}$$

Since by Proposition 10.2.4 $\lambda_{max} \geq \mid \lambda \mid$ and $\lambda_{max} \neq \lambda$ we have that $\lambda_{max} > \lambda$.
Let us consider two cases:

i) $\lambda < 0$
In the case of $\mathbf{x}^\lambda > \mathbf{0}$ we would have $\mathbf{A}\mathbf{x}^\lambda > \mathbf{0}$ and $\lambda \mathbf{x}^\lambda < \mathbf{0}$, clearly a contradiction with 10.60.

ii) $\lambda > 0$
Let us consider the left eigenvalue connected wit λ_{max}, $\dot{\mathbf{p}}$. By definition

$$\dot{\mathbf{p}}'\mathbf{A} = \lambda_{max}\dot{\mathbf{p}}' \tag{10.61}$$

Right-multiplying by $\mathbf{x}^\lambda$ and using 10.60 we have

$$\dot{\mathbf{p}}'\mathbf{A}\mathbf{x}^\lambda = \lambda_{max}\dot{\mathbf{p}}'\mathbf{x}^\lambda \tag{10.62}$$

$$\dot{\mathbf{p}}'\lambda\mathbf{x}^\lambda = \lambda_{max}\dot{\mathbf{p}}'\mathbf{x}^\lambda \tag{10.63}$$

$$(\lambda_{max} - \lambda)\dot{\mathbf{p}}'\mathbf{x}^\lambda = 0 \tag{10.64}$$

and since $\dot{\mathbf{p}}' > \mathbf{0}$ and $(\lambda_{max} - \lambda) > 0$, $\mathbf{x}^\lambda$ must have a negative component.

□

It can be shown that

Consideration V. $(\mathbf{I} + \mathbf{A})^{\mathbf{n}} = \mathbf{I} + n\mathbf{A} + \binom{n}{2}\mathbf{A^2} + ... + \binom{n}{n-1}\mathbf{A^{n-1}} + \mathbf{A^n}$

Consideration VI. Let $\mathbf{A}$ be a square matrix, λ_{max} its eigenvalue with the greatest absolute value, and $\mu > 0$ a positive real such that $\mu >| \lambda_{max} |$. Define also $\nu = \frac{1}{\mu}$. Then

$$(\mathbf{I} - \nu\mathbf{A})^{-1} = \sum_{i=0}^{\infty}(\nu\mathbf{A})^i \tag{10.65}$$

$$(\mu\mathbf{I} - \mathbf{A})^{-1} = (\frac{1}{\mu})\sum_{i=0}^{\infty}(\frac{1}{\mu}\mathbf{A})^i \tag{10.66}$$

Proposition 10.2.8. *Let $\boldsymbol{A}$ be a (n, n) non-negative, irreducible matrix and λ_{max} its maximum eigenvalue. Consider a positive real number $\mu = \frac{1}{\nu} > \lambda_{max}$. Then*

i) The following expression hold

$$(\mathbf{I} - \nu\mathbf{A})^{-1} > \mathbf{0} \tag{10.67}$$

$$(\mu\mathbf{I} - \mathbf{A})^{-1} > \mathbf{0} \tag{10.68}$$

ii) Each element of matrices $(\mathbf{I} - \nu\mathbf{A})^{-1}$ and $(\mu\mathbf{I} - \mathbf{A})^{-1}$ is a continuous, increasing function of ν and continuous and decreasing function of μ.

Proof. Consider the expressions in Consideration V and VI. Then

i) The first n elements of the infinite summations differ from the expression in Consideration V only because of coefficients ($1, n, \binom{n}{2}, ..., \binom{n}{n-1}, n$ in the first one, $1, v, ..., v^n$ in 10.65). But since all these coefiicients are positive numbers and from Lemma 10.2.1 $(\mathbf{I} + \mathbf{A})^{\mathbf{n}} > 0$, also the first n elements of 10.65 and 10.66 are strictly positive. Since the elements of 10.65 and 10.66 from $n + a$ to infinity are non-negative, *a fortiori* we have proved the proposition.

ii) It follows directly from Consideration VI

□

Proposition 10.2.9. *Let $\boldsymbol{A}$ be a (n, n) non-negative, irreducible matrix and λ_{max} its maximum eigenvalue. Let us denote with $\mathbf{a^i}$ the $i - th$ row of $\boldsymbol{A}$ and with $\mathbf{a_i}$ teh $i - th$ column of $\boldsymbol{A}$ and let $\mathbf{s}$ be the sum vector, composed of only ones. Then*

$$\max_i \mathbf{a^i s} \geq \lambda_{max} \geq \min_i \mathbf{a^i s} \tag{10.69}$$

$$\max_i \mathbf{s'a_i} \geq \lambda_{max} \geq \min_i \mathbf{s'a_i} \tag{10.70}$$

Proof. Let us divide teh demonstration in Two

i) Recall that λ_{max} is the maximum value of the function $\lambda(.)$ defined in 10.10. Consider $\lambda(\mathbf{s}) = \lambda_s$. By definition 10.10 we have that

$$\mathbf{As} \geq \lambda_s \mathbf{s} \tag{10.71}$$

$$\mathbf{a^i s} \geq \lambda_s \qquad for \quad i = 1, ..., n \tag{10.72}$$

This means that

$$\lambda_s = \min_i \mathbf{a^i s} \tag{10.73}$$

An since by definition $\lambda_{max} \geq \lambda_s$ we have

$$\lambda_{max} \geq \min_i \mathbf{s'a_i} \tag{10.74}$$

ii) consider now the matrix $(\lambda \mathbf{I} - \mathbf{A})$. Choosing a $\lambda \geq \max_i \mathbf{s'a_i}$ would result in the matrix being non-singular, with therefore a non-zero determinant. This means that λ could not be an eigenvalue of $\mathbf{A}$. We must have therefore

$$\max_i \mathbf{s'a_i} \geq \lambda_{max} \tag{10.75}$$

Notice that in the case of $\mathbf{a^i s} = \mathbf{a^j s} \forall i, j$ we would have

$$\max_i \mathbf{a^i s} = \lambda_{max} = \min_i \mathbf{a^i s} \tag{10.76}$$

Finally, notice that the demonstration for the 10.70 is analogous. □

10.3 Reducible Matrices

Let us now state the previous proposition withous the assumption of irreducibility.

Proposition 10.3.1. *Let λ_{max} be the positive max of function $\lambda(\mathbf{x})$. Let $\boldsymbol{A}$ be a (n, n) non-negative matrix. Then*

i) λ_{max} is an eigenvalue of $\boldsymbol{A}$, or

$$\mathbf{Ap} = \lambda_{max} \mathbf{p} \tag{10.77}$$

ii) To λ_{max} are associated non-negative right-eigenvector $\boldsymbol{p}$ and left-eigenvector $\dot{\mathbf{p}}$, or

$$\mathbf{p} \geq \mathbf{0}, \quad \dot{\mathbf{p}} \geq \mathbf{0} \tag{10.78}$$

Proposition 10.3.2. *Let $\boldsymbol{A}$ be a (n, n) non-negative matrix. Let λ be an eigenvalue of $\boldsymbol{A}$. Then*

$$\lambda_{max} \geq \mid \lambda \mid \tag{10.79}$$

(Note that it is identical to Proposition 10.2.4

Proposition 10.3.3. *Let $\boldsymbol{A}$ be a (n,n) non-negative matrix and λ_{max}^{A} its maximum eigenvalue. Then λ_{max} is a continuous, non-decreasing function of the coefficients of $\boldsymbol{A}$*

Corollary 10.3.3.1. *Let $\boldsymbol{A}$ be a (n,n) non-negative matrix and λ_{max}^{A} its maximum eigenvalue. Let $\mathbf{A_{ij}}$ be a square sub-matrix of $\boldsymbol{A}$. Then*

$$\lambda_{max}^{A} \geq \lambda_{max}^{A_{ij}} \tag{10.80}$$

Corollary 10.2.6.2 does not hold with irreducible matrices-

Proposition 10.3.4. *Does not exist a proposition analogous to Proposition 10.2.7 for reducibel martices*

Proposition 10.3.5. *Let $\boldsymbol{A}$ be a (n,n) non-negative and λ_{max} its maximum eigenvalue. Consider a positive real number $\mu = \frac{1}{\nu} > \lambda_{max}$. Then*

i) The following expression hold

$$(\mathbf{I} - \nu\mathbf{A})^{-1} \geq \mathbf{0} \tag{10.81}$$

$$(\mu\mathbf{I} - \mathbf{A})^{-1} \geq \mathbf{0} \tag{10.82}$$

ii) Each element of matrices $(\mathbf{I}-\nu\mathbf{A})^{-1}$ and $(\mu\mathbf{I}-\mathbf{A})^{-1}$ is a continuous, non-decreasing function of ν and continuous and non-increasing function of μ.

Proposition 10.3.6. *Let $\boldsymbol{A}$ be a (n,n) non-negative matrix and λ_{max} its maximum eigenvalue. Let us denote with $\mathbf{a^i}$ the $i-th$ row of $\boldsymbol{A}$ and with $\mathbf{a_i}$ teh $i-th$ column of $\boldsymbol{A}$ and let $\mathbf{s}$ be the sum vector, composed of only ones. Then*

$$\max_i \mathbf{a^i s} \geq \lambda_{max} \geq \min_i \mathbf{a^i s} \tag{10.83}$$

$$\max_i \mathbf{s'a_i} \geq \lambda_{max} \geq \min_i \mathbf{s'a_i} \tag{10.84}$$

(Notice that it is identical to Proposition 10.2.9)

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